THE MATHEMATICAL WORK OF JACQUES HADAMARD*

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It is not my intention to give here the biography of Jacques Hadamard; I shall not even try to outline it. Very fortunately he is living, living vigorously and still creating, and I hope he will give us mathematicians a good example of a very long life, even if we are unable to make ours as fertile as his. This month he had his 87th anniversary; this should give some of you a hint of the year when he was born.

Few mathematicians have ever worked in so many fields as did Hadamard, and when I say few, I mean almost none. When he retired as Professor of the Collège de France in 1937 after forty years of service, he had either taught almost every mathematical subject or participated very efficiently in discussions about it in his famous "Séminaires" at the Collège. I have never seen or heard about weekly mathematical events as important, as passionately interesting, as those "Séminaires."

Great mathematicians, and lesser ones too, from countries all over the world, mathematical countries, I mean, considered it a very great honor to be invited by Hadamard to speak about their own work, or about a paper recently published in the subject they knew best. I mean the subject they knew better than any other subject; but usually that same subject, whatever it was, and whoever spoke, was still better known, or at least better understood, by the Master, Professor Hadamard.

How often we youngsters (I speak of the twenties, when even some of *us* were young!), how often, I say, we youngsters felt embarrassed and blushed, and felt much pity for men with great names, sometimes very great names, who triumphantly announced fine and deep theorems, in fields which were their fortresses, when these theorems were, after their exposition, "explained" to the audience *and* to the lecturer, the specialist, by Hadamard—explained and related to other fields in mathematics, and very often simplified. In those seminars, ideas, techniques, knowledge were braced together, and from them, a few weeks later, sometimes on the spot, came new results. The great inspirer was Hadamard.

We shall only speak about ideas and results published in his own papers and books, ideas which often have completely changed the aspect of a field, and, even more often, created new fields.

I remember Hardy introducing Hadamard to the London Mathematical Society in 1944, where he was asked to speak about his most striking discoveries. Hardy called Hadamard the "living legend" in mathematics.

I shall do my best to bring before you some aspects of this legendary mathematical career.

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I said I will not even outline Hadamard's biography, but you may be curious to know that he was not a mathematical "*Wunderkind.*" As he says himself, until the age of eleven or twelve he was the worst, or almost the worst, student in arithmetic. He once asked his father, himself a Professor, if mathematics were studied at the Ecole Normale. "Yes, of course," his father said, "in the Science Section." "Oh, well," was young Hadamard's answer, "then it is not for me."

His first publications appeared in 1884 when he was nineteen. But not until 1888 were Notes published in the Comptes Rendus de l'Académie des Sciences which began to make him famous. Historically I have thus to begin with the Theory of Functions of a Complex Variable.

The Notes of 1888 and 1889 were an introduction to Hadamard's Thesis which was published in 1892, and which rapidly became classical.

That year of 1892 is one of the richest in the history of modern Function Theory, since, in that year, there not only appeared Hadamard's Thesis devoted mostly to the research on singularities of an analytic function by means of its Taylor coefficients at a point, but there also appeared his famous work on entire functions. His function theoretic results of 1892 enabled him also, a few years later, to solve one of the ancient and most important problems in the Theory of Numbers.

Let us now speak about ideas, theorems, results; we shall rarely mention years, data, titles. . . .

It had been known since the work of Abel and Cauchy that a Taylor series defines completely an analytic function, and, defining precisely the meaning of analytical continuation, Weierstrass, and Méray in France, took that series as the starting point for the definition of an analytic function. In other words, when the sequence of coefficients of an entire series is given, the corresponding function is unique. But it is merely an uniqueness theorem, and the problem arises of indicating the properties of the function so defined by the coefficients, for instance when the radius of convergence is finite. Few results of the inverse problem were known. For instance, the arithmetical nature of coefficients of an algebraical function were indicated by Eisenstein and Tchebycheff. Darboux has shown some properties of growth of coefficients when the singularities, supposed simple in nature, are known. An interesting, but false, result on the affixes of singularities, when the coefficients are known, was given by Lecornu.

But Hadamard is the real creator of the theory of detection of singularities of the analytical continuation of a Taylor series.

He, first, gave the value of the radius of convergence of a Taylor series, in introducing the notion of limit superior of the $|a_n|^{1/n}(f(z) = \sum a_n z^n)$. Until then only when the limit of this expression exists could one find the radius.

He gave then a necessary and sufficient condition bearing on the a_n in order that a given point on the circle of convergence be a singularity. This condition, given by expressions each of which contains a finite number of coefficients, has

since furnished, under its original form, or correspondingly adapted forms, a wealth of important results. And first of all, the famous "Hadamard's gap theorem": $\sum a_n z^{\lambda_n}$, if $\lambda_{n+1}/\lambda_n > \lambda > 1$, admits its circle of convergence (if finite) as a cut—is only a particular case of a result which follows from the criterion on singularities.

He also proved the infinitely elegant and important theorem on polar singularities related to the deep study of determinants formed by the coefficients, and discussed the classification of singularities on the circle of convergence by their "order," a notion rich in content, but which, to my understanding, was not fully used by mathematicians of later generations. It is in relationship with this notion of order that Hadamard introduced functions of "*écart fini*."

These are briefly, too briefly, I know, the results of Hadamard's Thesis. He got his degree! Some years later he proved his no less famous theorem on composition of singularities. Stated rapidly it reads: $\sum a_n b_n z^n$ has no other singularities than those which can be expressed as products of the form $\alpha\beta$ where α is a singularity of $\sum a_n z^n$, and where β is a singularity of $\sum b_n z^n$.

For sixty years these results have served as inspiration or as tools for a very great number of mathematicians. Few mathematical publications which have appeared since have had an influence on the minds of young mathematicians comparable to that of the two Hadamard papers I have just mentioned.

But let us come back to 1892. In that year Hadamard presented to the Academy a paper, on entire functions and on Riemann's ζ-Function, for the Grand Prix de l'Académie des Sciences. The mathematical world in Paris was sure that the great mathematician Stieltjes would get the prize. Stieltjes thought, indeed, that he had proved the famous "*Riemannische Vermutung*": the zeros of $\zeta(s)$ lie, all, on the line $\frac{1}{2}+it$. Let me quote a passage of Hadamard's famous paper (1896) on the Theory of Numbers: "Stieltjes has proved, as predicted by Riemann, that those zeros (of $\zeta(s)$) are all of the form $\frac{1}{2}+it$, but his proof was never published, and it was not even established that the function ζ has no zeros on the line $R(s)=1$."

But, it was Hadamard who won the prize, and I believe that you would have given it to him too. For, by means of the results of his thesis, he was able in his paper presented to the Academy, to show the relationship between the coefficients of the Taylor series of an entire function and its zeros; he could then also evaluate the genus of the entire function. These results had a dramatic bearing on the Theory of Numbers, since if ρ_n denote the zeros of $\zeta(s)$, $\sum 1/|\rho_n| = \infty$, and $\sum 1/|\rho_n|^2 < \infty$. In the paper, from which I quoted a passage a few minutes ago, he also proved that ζ does not take the value zero on $\sigma=1$, and thus he had now everything he needed to prove the most important proposition on the behavior of primes.

Here, I believe, we should give some older historical facts.

At the beginning of the last century Legendre made the assumption that, on denoting by $\pi(x)$ the number of primes smaller than x (I use, of course,

modern notations), then

$$\pi(x) = \frac{x}{\log x - A(x)}$$

with $A(x)$ tending to a finite limit.

From this it should follow that $\pi(x) \sim x/\log x$. This remained the great unsolved assumption for almost exactly a century.

It is true that Tchebycheff, in the middle of the nineteenth century, by very fine and difficult technical means, had shown that

$$.92129 \leqq \underline{\lim} \frac{\pi(x) \log x}{x} \leqq \overline{\lim} \frac{\pi(x) \log x}{x} \leqq 1.10555 \cdots,$$

but he did not prove that $\pi(x) \log x/x$ tends to a limit, and there was no hope that his method would be able to furnish such a proof. Many mathematicians were able, it is true, to tighten the limits of indetermination indicated by Tchebycheff, on using the same methods, perhaps somewhat improved. Sylvester was among them. But there was nothing fundamentally new in these improvements. Let us quote Sylvester (1881) on this matter: "But to pronounce with certainty upon the existence of such possibility ($\lim \pi(x) \log x/x = 1$) we shall probably have to wait until someone is born into the world as far surpassing Tchebycheff in insight and penetration as Tchebycheff proved himself superior in these qualities to the ordinary run of mankind."

And, as Landau says, when Sylvester wrote these words Hadamard was already born (I imagine that by now you have calculated the year of his birth!)

Riemann introduced his ζ function for the study of the behavior of primes. But he did not prove Legendre's assumption; he did, of course, prove important properties of the function; he made important assumptions on it which were proved to be right and which helped a lot. (I do not speak here about *the* assumption: the "*Vermutung.*")

Hadamard, based on his own result on the genus of an entire function related to ζ, a result made possible only because of his fundamental papers on analytic functions, quoted a little while ago, and based also on his proof that $\zeta(s) \neq 0$ for $\sigma = 1$, has given the complete proof of the great assumption on the distribution of primes.

He also proved analogous theorems on the distribution of primes belonging to a given arithmetical progression since by his methods he was able to study Dirichlet series which, with respect to those primes, play the same role as ζ plays with respect to all the primes.

I must not leave the Theory of Functions without mentioning at least some facts which play such an important role in modern mathematics; let us mention only the titles: The three circle theorem which showed the importance of convexity in the study of analytic functions; the statement of the problem of quasi-analyticity, as distinct from the problem of generalizing the notion of analytic

functions in the complex domain; and, since I have spoken above about determinants, Hadamard's determinants, let us not forget to recall his very important estimate of the value of a general determinant.

We should like now to speak of a paper of Hadamard related to topology, to Analysis Situs, to use the expression of that time, which influenced very much some eminent American mathematicians. That paper dealt with the geodesics on surfaces with opposite curvatures. As you see, the title has a geometrical sound. Was it this paper, or rather the title of this paper, which was almost the cause of a tragedy? Well, the word is perhaps too strong, but here is the story.

The period in which it happened was really a tragic one for France: the period of the Dreyfus trial. Nobody was neutral; everybody was either for Dreyfus or against him. Hadamard was, of course, very much of a "Dreyfusard" (that is to say, he believed in the innocence of Dreyfus). The great Hermite was rather on the other side. It was also a period when people in France were very polite, a period when young scientists still respected the old ones and did not mind showing it. At New Year's they used to pay a visit to their "Maîtres" to wish them a Happy New Year. So Hadamard, the young Hadamard, went to pay his respects to the venerable Hermite. But when Hermite saw the young man he exclaimed: "Hadamard, you are a traitor!" If you remember what such a compliment meant during Dreyfus' trial, and especially addressed to a man like Hadamard, you will understand Hadamard's reactions. Emotion, anger—"Yes, Hadamard, you are a traitor, you have betrayed Analysis for Geometry!"

On the surfaces of negative curvature considered by Hadamard, surfaces admitting a finite number of expanding infinite nappes, the geodesics behave in a way which presents an interesting philosophical problem to physicists and astronomers.

There are 1) closed geodesics, or geodesics asymptotic to such closed ones; 2) there are geodesics which tend to infinity on any of the nappes. But there is also a third category of geodesics on these surfaces, those of which entire segments approach successively corresponding segments on each of a sequence of closed geodesics, the length of these segments increasing constantly.

But the most remarkable feature is the following: the set E of tangents to the geodesics passing through a point O and remaining at a finite distance is perfect, nowhere dense. As a matter of fact, in the neighborhood of each geodesic of which the tangent belongs to E (we speak of course of neighborhoods of directions), there is a geodesic which goes to infinity in an arbitrarily chosen nappe. In each such neighborhood there are also geodesics of the third category, *i.e.*, those which approach ever more tightly a denumerable set of closed geodesics.

In a problem in physics a slight modification in the data at a moment should influence little the future of a solution, since only approximate data are known anyway. If a problem of trajectories in a question in astronomy presents the

same feature as that of the geodesics of surfaces studied by Hadamard, one could then foresee the philosophical difficulties he will have to solve.

Hadamard's work on equations with partial derivatives furnished one of the greatest contributions to this field during the first part of the century: the study of the possibility of a solution of a problem in relation to its physical character, the fundamental difference between Cauchy's and Dirichlet's problems from that point of view.

In a general way the entire approach to the question involved depends on the nature of the characteristics and the choice of initial conditions. His greatest achievement was perhaps the systematic introduction of convenient fundamental solutions.

Hadamard applied these ideas to dynamics in the motion of waves, having first made a general study of the kinematics of continuous media. One of the most inspiring pieces of work is his study of the principle of Huyghens; he could point out the cases where the principle really appears. This is for instance the case when the given system comes into rest after the passage of a wave; when the principle does not hold it corresponds to the case where after the passage of the wave a residual wave remains.

I will only mention by name his work on integro-differential equations, first introduced by him, his classical formula on the variation of Green's function corresponding to a small variation of the corresponding region, his formula, expressed by a limit, of a linear functional. He was the first man to envisage the Calculus of Variations by means of functionals of Volterra.

THE CONVERGENCE OF NUMERICAL ITERATION*

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Iteration arises frequently in the numerical solution of applied problems. Textbook statements on its convergence are often cursory. We hope this note will put students on their guard.

We shall deal with solving

$$x = f(x) \tag{1}$$

by means of the iteration

$$x_n = f(x_{n-1}) \tag{2}$$

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starting with a trial value x_0. For simplicity we shall suppose throughout that $f(x)$ is a real function of a real variable x, that (1) possesses a unique solution, and that this solution is $x=0$. From the point of view of theory, there is no real loss of generality involved in this last assumption; for if $x=a\neq 0$ were the solution of $x=f(x)$, then $x=0$ would be the solution of $x=g(x)=f(x+a)-a$. To avoid triviality, we shall also suppose throughout that the initial value x_0 is not zero.

We shall confine our attention to equations in a single unknown x; the difficulties in the case of several unknowns are more severe.

Under certain conditions upon the function $f(x)$ the sequence $x_0, x_1, x_2, \cdots$ will converge to the desired root $x=0$. We now invite the reader, who cares to test his appreciation of such conditions, to answer the following questions (with the aid of a textbook if he so desires).

Question 1: Is it sufficient for convergence that, for some given $k<1$, $|f'(\xi)| \leqq k$ for every ξ in a (sufficiently small) neighborhood of the root $x=0$, and that x_0 shall belong to this neighborhood? (In this question and subsequently, $f'(x)$ denotes the derivative of $f(x)$.)

Question 2: Is it sufficient for divergence that, for some given $k>1$, $|f'(\xi)| \geqq k$ for every ξ in a (sufficiently small) neighborhood of the root $x=0$, and that x_0 shall belong to this neighborhood?

Question 3: Consider two functions $f_1(x)$ and $f_2(x)$ which both satisfy the conditions of Question 1 in a *common* (sufficiently small) neighborhood of the root $x=0$; and suppose that k_1 and k_2 are respectively the smallest possible values of k for which these conditions hold [*i.e.* there exist no constants $k_1' <k_1$, $k_2' <k_2$ such that $|f_1'(\xi)| \leqq k_1'$, $|f_2'(\xi)| \leqq k_2'$ for all ξ in this neighborhood]. If $k_1<k_2$ and if both iterative processes converge, will the convergence for $f_1(x)$ be more rapid than that for $f_2(x)$?

Question 4: Is it necessary for convergence that all conditions stated in Question 1 shall hold (a) if we restrict ourselves to the class of functions $f(x)$ which are *everywhere* differentiable or (b) if we make no such restrictions?

Question 5: Can a condition bearing on the derivative $f'(0)$ or on a Lipschitz condition solely at the root $x=0$ be sufficient for convergence?

Question 6: Are there any functions $f(x)$, satisfying $f(0)=0$ and being discontinuous both to the left and to the right of $x=0$, for which the iteration (2) converges whenever x_0 lies within some (sufficiently small) neighborhood of $x=0$?

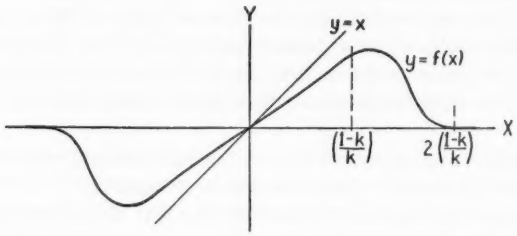
The answer to Question 1 is "Yes." This is a corollary to our answer to Question 5 (see below).

The answer to Question 2 is "No" as shown by the example

$$f(x) = \begin{cases} 2x & |x| \leqq 1 \\ 0 & |x| > 1 \end{cases} \tag{3}$$

for which the process (2) converges.

The answer to Question 3 is "No." The example



$$(4)\qquad f(x)=\begin{cases}(\tfrac{3}{2})^2kx & |x|\leqq(1-k)/k\\ (\tfrac{3}{2})^2(kx+2k-2)^2(3kx+2k-2)/(1-k)^2 & \\ & (1-k)/k<|x|<2(1-k)/k\\ 0 & |x|\geqq 2(1-k)/k\end{cases}$$

satisfies the conditions of Question 1; and yet just one step of the iteration will yield the desired root $x=0$ if

$$k\geqq 1/(1+\tfrac{1}{2}|x_0|);$$

whereas an infinite number of steps is needed if

$$k<1/(1+\tfrac{1}{2}|x_0|).$$

The convergence is slowest in the case

$$k=1/\left(1+\frac{9}{10}|x_0|\right).$$

The smallness of k, therefore, is not always a guarantee for rapid convergence as stated in textbooks. In practice, however, it will be a useful (though not wholly reliable) guide for the rapidity of convergence.

The answer to Question 4(a) is "No." The process (2) converges (very rapidly) for the function

$$(5)\qquad f(x)=\begin{cases}|x|^{3/2}e^{-x^2}\sin\left(\dfrac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$$

whatever the initial value x_0; although $f'(x)$ is unbounded in every neighborhood of the root $x=0$. *A fortiori*, the answer to Question 4(b) is "No." In one of the standard textbooks the conditions of Question 1 are falsely stated as both *necessary* and sufficient for convergence.

The answer to Question 5 is "Yes." It is sufficient for convergence that, solely at the root $x=0$, $f(x)$ shall satisfy a Lipschitz condition (of order unity) with an implied constant k less than unity: that is to say

$$\limsup_{x\to 0}\left|\frac{f(x)}{x}\right| \leqq k < 1. \tag{6}$$

For, if (6) holds, there exists a number $\delta>0$ such that

$$|x_n| = |f(x_{n-1})| \leqq K|x_{n-1}|, \qquad k < K < 1$$

whenever $|x_{n-1}| \leqq \delta$. Therefore, if x_0 belongs to the neighborhood $|x| \leqq \delta$, $|x_n| \leqq K^n|x_0| \to 0$ and $n \to \infty$. The reader will see that $f(x)$ can satisfy (6) even though it may not be differentiable; if, however, $f(x)$ is differentiable, we may replace (6) by the condition

$$|f'(0)| < 1. \tag{7}$$

Condition (7) is a weaker condition than the condition of Question 1. The reader will also notice that the foregoing proof of convergence is very much shorter than the corresponding proofs of weaker versions found in some textbooks. Finally, a condition sufficient for convergence need hold only at a single point, as we have just shown; but an analogous condition sufficient for divergence would, it seems, have to hold for all points; see for instance example (3) above.

The answer to Question 6 is "Yes." A simple example is

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Here at most two steps yield the root $x=0$, although the function is everywhere discontinuous. A more elaborate example shows that convergence is still possible even though $f(x)$ is everywhere discontinuous, everywhere unbounded, and non-measurable. For let I denote the set of all irrational numbers which are not rational multiples of π; and let S be a non-measurable subset of I. Put $f(x)=0$ if x is rational, $f(x)=\frac{1}{2}$ if x belongs to S, $f(x)=q$ if $x=p\pi/q$ where p/q is a non-zero rational number in its lowest terms, and $f(x)=1$ otherwise. Again at most two steps yield the root $x=0$. This example also suggests that the discovery of conditions necessary for convergence may be difficult or impossible.

It is only fair to add that the foregoing discussion is theoretical and academic. In practice, it will not be known that the root lies at $x=0$; it may merely be that we know the root lies between 0 and 1, say. In such a case, the comforting condition for convergence will be that $|f'(x)| \leqq k<1$ for all x in $0 \leqq x \leqq 1$.

UPPER LIMITS TO THE REAL ROOTS OF POLYNOMIAL EQUATIONS

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1. Introduction. In the solution of a numerical polynomial equation with real coefficients, it is important that the approximate location of the real roots be known. In particular, we should like to know beyond what limits no roots may exist. To some extent, calculations involved in the application of Horner's or Newton's process or the method of iteration may be shortened by utilizing these bounds.

This problem first received attention in the seventeenth century. In his *De Limitibus Aequationum*, Florimond de Beaune (1601–1652), a distinguished French mathematician and successor of Descartes, endeavored to show that the limits of positive roots might be found from the coefficients in equations up to the fourth degree. Unfortunately, he merely considered specific cases and did not venture to solve the problem for the n-th degree. As early as 1683, John Wallis, in commenting on de Beaune's and also on E. Bartholinus's rules for finding bounds, stated that this "subject is yet capable of further development" [17]. The first general methods for obtaining limits date from Newton (1722) and Maclaurin (1748). Since then, researches of many mathematicians have resulted in numerous methods for obtaining more useful bounds. The labor involved in applying some of these procedures, however, causes them to be of theoretical interest only. In this paper, the more important rules will be described and compared as to relative accuracy and efficiency.

Throughout, theorems have been translated to conform with the following notation. Let:

(1) $f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, $a_0 > 0$;
(2) x_i = any positive real root of $f(x) = 0$;
(3) s = the largest x_i ($s = 0$, if no x_i exists);
(4) k = the number of non-negative coefficients preceding the first negative one;
(5) b = the least a_i, $i = 1, \cdots, k-1$;
(6) d = the greatest a_i, $i = 0, \cdots, k-1$;
(7) g = the absolute value of the greatest negative coefficient;
(8) G = the absolute value of the second greatest negative coefficient;
(9) c = the number of negative coefficients. (If $c = 0$, then $s = 0$, and g and G need not be defined.)

Each of the cited methods provides a rule for locating a non-negative value of x satisfying $f(x) \geqq 0$, together with the added requirement that the substitution for x of all greater positive numbers yields this same inequality.

For greater clarity, the various procedures are grouped under separate subdivisions. Observed likenesses of form and approach led to these arbitrary classes. Thus the proofs of all tests in type I involve the given function and its

derivatives. The rules listed under II all seem to be very flexible in numerical situations. All formulae in III reduce to the same "set" form under certain conditions. Similarity of form led to IV, while Sylvester's process was placed in a different category.

2. Type I. The basic theorem here is Newton's rule:

I:A. *If*

$$f(x_0), f'(x_0), f''(x_0), \cdots, f^{(n)}(x_0)$$

are non-negative, then $s \leqq x_0$. [15]

Very close answers are obtained when the equation, $f(x)=0$, has only real roots. Presence of any complex roots may tend to augment the calculated upper limit. Practically speaking, however, the theorem is too cumbersome.

Designed to cut down on the labor entailed in using I:A is Laguerre's procedure:

I:B. *If* x_0 *is non-negative, and if all the coefficients in the quotient and remainder of* $f(x)$ *synthetically divided by* $(x-x_0)$ *are non-negative, then* $s \leqq x_0$. [16]

While this test does reduce the numerical work in I:A, it can come no closer to the greatest x_i. Efforts at improving I:B usually increase the computations.

The following two processes are more useful when dealing with tabular functions rather than with polynomial equations.

I:C. *If for some positive* $\Delta x = h$,

$$f(x_0), \Delta f(x_0), \Delta^2 f(x_0), \cdots, \Delta^n f(x_0),$$

are all positive, then

$$s < x_0 + (n-1)h. \quad [14]$$

I:D. *If*

$$f(x_0), \Delta f(x_0 - h), \Delta^2 f(x_0 - 2h), \cdots, \Delta^n f(x_0 - nh)$$

are all positive, then $s < x_0$. [14]

The connection between the last cited theorems and Newton's rule can be seen by applying the fundamental relationship between finite differences and derivatives to Newton's formula for forward interpolation. The Taylor expansion and consequently I:A will result.

3. Type II. Among the different procedures here classified is the following elementary one:

II:A. *Using the inequality* $f(x) \geqq 0$, *if the negative terms of the polynomial are transposed to the right of the inequality sign and the entire inequality is divided by* x^n, *then* $s \leqq x_0$, *where* x_0 *is any non-negative value of* x, *for which, and for all higher values, the assumed inequality holds.* [18]

Even though this is but a rewording of the definition for upper limit, it may yield a closer bound than some more formalized test.

Another rule to be noted is this fundamental one:

II:B. *If the terms of an equation are arranged in groups wherein the coefficient of the highest power of x appearing is positive, s will equal or be less than that non-negative value of x for which each group is non-negative.* [2]

Ingenuity is needed to get a reasonably useful answer. For example, consider the equation

$$x^4 - 4x^3 + 33x^2 - 2x + 18 = 0.$$

Here, a regrouping such as

$$x^3(x - 4) + x(33x - 2) + 18 = 0,$$

gives $s<4$. Yet a rearrangement such as

$$x^2(x^2 - 4x + 5) + (28x^2 - 2x + 18) = 0$$

yields $s=0$ since both quadratic groups have only complex roots. Thus, insight seems to be a major factor in success.

A third method which may supply varying results with regard to one equation is as follows:

II:C. *To simplify solution, modify the given equation in any way provided that the coefficient of the highest degree term remaining be positive and that the value of $f(x)$ for $x \geqq 1$ be not increased. If s^* denotes the largest x_i of the new equation,*

$$s < \max [1, s^*]. \text{ [7]}$$

Typifying one objection against this rule's use is the case of

$$x^5 - x^4 - x^3 - 2x^2 - 3x - 256 = 0.$$

The effort spent in manipulation might more profitably be expended employing another procedure or even approximating to the roots.

4. Type III. At the other extreme of flexibility are the "set" formulas which rely chiefly on the magnitude of coefficients present. The most commonly cited seem to be:

III:A. $s<1+(g/a_0)^{1/k}$. [5]

III:B. *If each negative coefficient be taken positively and divided by the sum of all preceding positive coefficients, s will be less than the greatest quotient thus formed increased by unity.* [2]

The usefulness of these theorems stems from the fact that both give relatively close bounds without much computation.

The remaining rules in this class are basically similar since all attempt to

improve on III:A and III:B either by allowing the other coefficients to have weight or by lessening the complexity involved. Usually, progress made in one direction is achieved at the other's expense.

Among the simpler tests, we have Maclaurin's rule:

III:B:1. $s<1+g/a_0$. [13]

The sole advantage which this theorem has over the preceding ones is its ease in application. Although sometimes both III:A and III:B reduce to III:B:1, in general method III:B:1 yields useless results.

A modification proposed by Vene in 1822 is:

III:B:2. $s<1+g/d$. [18]

This procedure can be of use only when there are several relatively large positive coefficients preceding the first negative one.

Some other methods embodying but a minor variation in form are:

III:B:3. $s<1+g/\sum_{j=0}^{k-1}a_j$. [8]

III:B:4. $s<1+(g+G)/(2a_0+2a_1+\cdots+2a_{k-2}+a_{k-1})$. [8]

III:C:1. $s<(1+g/b)^{1/(k-1)}$, $b>0$. [13]

III:C:2. $s<(1+g/a_0)^{1/k}$. [13]

III:D. $s<(1+g/a_p)^{1/(k-p)}$, where $a_p(\neq 0)$ *is the coefficient of the* $(p+1)$*th term, and* $p<k$. [10; also §8A below]

Let us now consider some "set" rules of greater complexity, the first being:

III:C. *If an arbitrary number of such pairs of terms* $a_{n-j}x^j$ *and* $a_{n-j+A}x^{j-A}$ *with* $a_{n-j}\geqq|a_{n-j+A}|$, $a_{n-j+A}<0$, $A<j$, *are dropped and if, in the resulting equation* $a_{n-m}x^m$, *where* $a_{n-m}\neq 0$, *is any term preceding all negative ones and* $(m-K)$ *is the power of the first negative term, then in sequential form*:

$$s\leqq 1+(g/a_{n-m})^{1/K};\ \phi[1+(g/a_{n-m})^{1/K}];\cdots$$

where $\phi(x)\equiv 1+[(1-x^{(-m+K-1)})g/a_{n-m}]^{1/K}$; $x>1$. [7]

When no significant changes can be made in the equation, the first term of the bounding sequence will equal III:A. By continuing the iterative process, one may approach somewhat closer to s. Sometimes the sequence may converge on s, thus providing a very limited process for approximating to an incommensurable root. Wherever the equation can be modified extensively, this procedure may prove more useful.

The following, a very complicated rule, reduces to simpler forms in certain cases:

III:F. *Given any number* h *satisfying the inequality*

$$g - (h-1)\sum_{j=0}^{k-1} a_j h^{(k-1)-i} > 0,$$

then

$$s < \max\left\{h,\ 1 + \left[\left(g + a_0(h-1)^k - (h-1)\sum_{j=0}^{k-1} a_j h^{(k-1)-i}\right)\Big/ a_0\right]^{1/k}\right\}. \quad [4]$$

Taking $k=1$, one obtains Maclaurin's rule. If $h=1$, method III:A results. Best use is made of the theorem when a relatively large difference exists between

$$a_0(h-1)^k \quad \text{and} \quad (h-1)\sum_{j=0}^{k-1} a_j h^{(k-1)-i}.$$

From a theorem stated incorrectly by Corliss [4; also §8C below] two rules can be derived:

III:G:1. $s<\max\left\{2,\ 1+\left[\left(g-\sum_{j=0}^{k-1} a_j\right)\Big/ a_0\right]^{1/k}\right\}$ *whenever the radicand* >1 *and* $k>1$.

III:G:2. $s<\max\left\{2,\ 1+\left[\left(g+1-\sum_{j=0}^{k-1} a_j\right)\Big/ a_0\right]^{1/k}\right\}$ *whenever the radicand* >1.

Of these two variations, the first, where applicable, will supply a somewhat more useful result.

Still greater formality is found in Mourgues's inaccurately described theorem [9, §8D, below]:

III:H. *If the negative terms of an equation are so taken that the exponents of the variable* x:

$$n-k,\ n-q,\ \cdots,\ n-r,\ n-t$$

form a decreasing sequence of positive integral numbers and the corresponding coefficients are so modified that

$$a_k/a_0 = A_k;\ a_q/a_0 \leqq Q;\ \cdots;\ a_r/a_0 \leqq R;\ a_t/a_0 \leqq T$$

where

$$A_k \leqq Q \leqq \cdots \leqq R \leqq T,$$

then, given any number $H \geqq 2$,

$$s < \max\left\{1 + \left[(A_k H^{t-k} + QH^{t-q} + \cdots + RH^{t-r} + T)/(H^{t-k} + H^{t-q} + \cdots + H^{t-r} + 1) - \sum_{j=1}^{k-1} (a_j H^{k-j-1})/a_0\right]^{1/k},\ H\right\}.$$

Unlike other procedures of this class, the above will reduce to III:B:1 when $k=1$ only if no negative coefficient exceeds a_1. Thus the method, because of its cumbersome form, may provide a very useful result.

In general, most of the tests in this section are much too inflexible. It seems best to rely on the two standard formulae, III:A and III:B, since many of the refinements are often nullified in numerical situations.

5. Type IV. The tests in this section are closely allied with those of preceding subdivisions. Nevertheless, in form, they lie in a separate grouping.

IV:A. $s \leqq \max\left[\{(c|a_i|)/a_0\}^{1/i}\right]$, *where* a_i *denotes negative coefficients only.* [3]

IV:B. $s < \max\left[\{|a_i|/a_0\}^{1/i} + \{|a_j|/a_0\}^{1/j}\right]$, $i \neq j$, *where* a_i *and* a_j *both denote negative coefficients only.* [11]

IV:C. $s < \max\left[2\{|a_i|/a_0\}^{1/i}\right]$, $i = 1, \cdots, n$. [1]

Usually IV:C gives the poorest values since it is not only dependent on the various negative coefficients but also on the positive ones. If n is relatively large and the negative terms occur quite near the beginning of the equation, it is better to use IV:B.

6. Type V. Sylvester's approach involves a simple continued fraction expansion of the quotient of two polynomials:

V. *Let* $$\frac{f'(x)}{f(x)} = 0 + \cfrac{1}{X_1 + \cfrac{1}{X_2 + \cdots}} = (0, X_1, \cdots, X_m),$$ *where* $X_i = A_i x - B_i$; $i = 1, \cdots, m$.

Then

$$s < \max\left[(B_1 \pm 1)/A_1;\ (B_2 \pm 2)/A_2;\ \cdots;\ (B_{m-1} \pm 2)/A_{m-1};\ (B_m \pm 1)/A_m\right]. \quad [12]$$

This method has never attained any but theoretical popularity because of the calculational difficulties arising in numerical cases. So far, any effort at modification to lessen the computation has only succeeded in increasing the limit obtained.

7. Summary. No startingly new conclusions can be drawn as to the relative merits of all these tests. It seems clear that generally some method of type I, III or IV should give a satisfactory result. In numerical examples, the possibilities of the more flexible forms in II should never be forgotten.

Thus, one ought to be aware of the major approaches. In a given situation, use should be made of the most applicable rule.

8. Appendix on Erroneous Statements. In the course of this study, the following erroneous or misleading remarks were noted:

A. By neglecting to note the significance of discarding a positive quantity, Myers [10] was led to assert that s might sometimes be achieved from III:D. In brief, he states:

"With regard to $x>0$,

(1) $$f(x) \geqq [a_p(x-1)x^{n-p} - gx^{n-k+1} + g]/(x-1).$$

Dropping g from the right-hand member does not affect the inequality when $x>1$. Thus,

(2) $$f(x) \geqq \{x^{n-p}[a_p(x-1) - gx^{p-k+1}]\}/(x-1), \text{ etc.}"$$

As the remainder of the proof is correct, it need not concern us here. The equality sign holds in (1) if $k=1$ and $a_i=g$ $(i=1, \cdots, n)$. It is tacitly assumed that $g>0$ since otherwise $s=0$. By dropping the positive g, we find that the equality will be affected. Therefore s can never be thus attained.

B. Here, too, should be cited Corliss's attempt [4] at adapting III:B to a special situation:

"If $k=1$, it is more effective to let $x=my$ and then to use III:B on the y-equation. Choose an m (either by inspection or by application of the theory involving maxima and minima) which will give a satisfactory upper limit."

Misleading results may arise, however, if one tries to apply this method to various equations with $a_1<0$. From a graphical standpoint, the most unfavorable case actually reduces to finding the minimum point on the curve:

$$x = \max\,[m + |a_1|/a_0; \cdots; m + |a_n|/a_0m^{n-1}],$$

where a_i $(i=1, \cdots, n)$ denote all negative coefficients. Unless a_1/a_0 is the greatest negative coefficient, one can not state definitely that the minimum point of the greatest x_i-curve $[x_i=m+|a_i|/a_0m^{i-1}]$ will coincide with that of the x-curve. Often, there is no quick way of determining the desired point. Thus, this method can be utilized to advantage only if after obtaining a value for m, that value is reinserted in the fractions a_i/a_0m^i, to check whether the original relationships still persist. If the originally picked fraction is at least one of the greatest (there being one or more having the same value), then only will the limit be correct.

C. At times erroneous is theorem III:G as given originally by Corliss [4]:

$$"s < \max\left\{2,\ 1 + \left[\left(g - \sum_{j=0}^{k-1} a_j\right)/a_0\right]^{1/k}\right\},\ \textit{whenever the radicand} > 1."$$

When $k=1$, the test may yield a number less than s. To discover why such an error can occur, it is necessary to examine the proof more closely:

"No $x>1$ will be a root of $f(x)$ if

(1) $$(a_0x^{k-1} + a_1x^{k-2} + \cdots + a_{k-1})(x-1) - g > 0.$$

Make use of the relationship

(2) $$x^{k-1} \geqq h^{k-1} - (h-1)^{k-1} + (x-1)^{k-1}$$

which is true when $x \geqq h \geqq 1$. Substituting for x^{k-1} and for x in (1), obtain:

$$(3) \qquad [(a_0h^{k-1} + \cdots + a_{k-1}) - a_0(h-1)^{k-1}](h-1) + a_0(x-1)^k - g > 0.$$

Putting $h=1$ in the bracket of (3) and $h=2$ in the multiplier, $(h-1)$, gives theorem III:G."

Now if $k=1$, expression (2) reduces to the triviality: $1=1$. Therefore, from (1) we have

$$s < 1 + g/a_0$$

which is just III:B:1. To alter III:G so that it can be used in all instances, give the value 1 to the term $a_0(h-1)^{k-1}$ occurring in (3). This is sufficient since 1 is the largest value which this term can achieve. So, the two modified forms, III:G:1 and III:G:2, result.

D. In discussing the use of III:G, Mourgues asserts [9] that one should begin by taking $H=2$, or 3. Then, if the limit obtained so be too great, one should replace H by one of the numbers 4, 5, $\cdots$. Unfortunately, he fails to mention that though H might be increased without limit, s is less than the greater of the two numbers, H and the calculated bound.

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MATHEMATICAL NOTES

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A PENTAGON THEOREM

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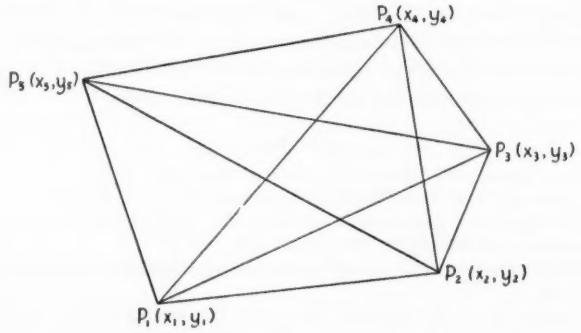
Let $P_i(x_i, y_i)$, $(i=1, \cdots, 5)$ be any five points in a plane, referred to a system of rectangular axes, no three of which lie in a straight line. Connect them as shown in the figure below. Consider the quadrilateral $p_1p_2p_3p_4$. Let $L_i=0$, $(i=1, \cdots, 4)$,



be the equations of its sides, where (x_i, y_i) is the initial point of the side, reading in a counter-clockwise direction. Then,

$$\frac{L_1 \cdot L_3}{L_2 \cdot L_4} = \lambda \tag{1}$$

is the equation of a system of conics through P_1, P_2, P_3, and P_4. If the equations of the sides $L_i=0$ are expressed in determinant form and if (x_5, y_5) is regarded as the variable point, (1) may be written in the form

$$\frac{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_5 & y_5 & 1 \end{vmatrix} \begin{vmatrix} x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}}{\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \end{vmatrix} \begin{vmatrix} x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}} = \lambda_5 \tag{2}$$

where the subscript on λ indicates the variable point.

Each determinant in (2) is twice the area of a certain one of the four triangles of the pentagon formed by $P_1, \cdots, P_5$. Let these triangular areas be represented by D with subscripts indicating the points involved. Equation (2), then, takes the form

$$\frac{D_{125}D_{345}}{D_{235}D_{145}} = \lambda_5.$$

Four additional similar relations may be formed from the other four quadrilaterals of the pentagon, all of which may be expressed by the equations,

$$\lambda_i = \frac{D_{i,i+1,i+2}D_{i,i+3,i+4}}{D_{i,i+1,i+4}D_{i,i+2,i+3}}, \qquad (i = 1, \cdots, 5), \tag{3}$$

where it is understood here and hereafter that the subscripts 6, 7, 8 and 9 are to be replaced, respectively, by 1, 2, 3, and 4.

In a paper by the author, *Permanent configurations in the problem of five bodies*,* it was shown that if the D's are eliminated from (3), the λ's satisfy a system of equations of the form

$$\lambda_i - \lambda_{i+2}\lambda_{i+3} - \lambda_i\lambda_{i+2}\lambda_{i+3} = 0, \qquad (i = 1, \cdots, 5),$$

which, by using (3), may be expressed in the equivalent form

$$D_{i,i+1,i+3}D_{i,i+2,i+4} = D_{i,i+1,i+2}D_{i,i+3,i+4} + D_{i,i+1,i+4}D_{i,i+2,i+3}, \qquad (i = 1, \cdots, 5), \tag{4}$$

a system of five equations which are identically satisfied by any pentagon whatever, either concave or convex.

Consider any vertex P_i, $(i=1, \cdots, 5)$, of the pentagon and the quadrilateral formed by the remaining vertices. Call this quadrilateral the *associated* quadrilateral of P_i. Let the triangles of the pentagon having P_i as one vertex and a diagonal of the associated quadrilateral as a side be called a *diagonal* triangle and call the other triangles of the pentagon formed by P_i and the sides of the associated quadrilateral *side* triangles. Then by equations (4) we have the THEOREM: *In any pentagon the product of twice the areas of any two diagonal triangles belonging to the same quadrilateral is equal to the sum of the products of twice the areas of the opposite side triangles, where in forming the triangular areas the vertices of the pentagon are to be taken in the order P_1, P_2, P_3, P_4, and P_5.* It may be of interest to observe that this theorem bears some resemblance to the theorem of Ptolemy which states that if a quadrilateral is inscribed in a circle the product of its diagonals is equal to the sum of the products of the opposite sides. It could be regarded, perhaps, as the companion theorem to Ptolemy's, for a pentagon inscribed in a conic.

* Trans. Amer. Math. Soc., vol. 44, No. 3, November 1938, pp. 563–579.

ON A THEOREM OF BÉLA SZ.-NAGY

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If M and N are subsets of a metric space and f maps M onto N, then f will be called "Lipschitzian" provided

$$\sup \{\rho(fx, fy)/\rho(x, y): x, y \in M, x \neq y\} < \infty.$$

Béla Sz.-Nagy observed that if M is the unit cell $\{x: \|x\| \leqq 1\}$ in Euclidean k-space, N is a bounded closed convex body whose interior includes the origin $\varnothing$, and f is the radial projection (defined below) of M onto N, then f is Lipschitzian (his term: "dehnungsbeschränkt"). A proof has been given by Vincze and Szüsz [3]. Now in terms of the gauge functionals of the sets, the radial projection can be defined quite simply (see below). The purpose of this note is to show how the result of Sz.-Nagy follows (in an arbitrary normed linear space) from elementary properties of gauge functionals.

By a "gauge" on a real linear space L is meant a real-valued function g on L which satisfies the following conditions: (1) $g(tx) = t(gx)$ whenever $t \geqq 0$; (2) $g(x+y) \leqq gx+gy$; (3) $gx > 0$ whenever $x \neq \varnothing$. The fundamental relationship between gauges and convex sets is well-known [1; pp. 21–22]: If g is a gauge, then the set $C_g = \{x: gx \leqq 1\}$ is convex and intersects each line through $\varnothing$ in a closed segment having $\varnothing$ as an inner point. Conversely, to each such set C corresponds a unique gauge $g = g_c$ such that $C_g = C$. This gauge can be defined by: $g_c x = \inf \{t: t > 0, t^{-1}x \in C\}$.

Suppose B and C are bounded closed convex bodies in a normed linear space, each including $\varnothing$ in its interior. Let FB and FC be the boundaries of B and C. For each ray ρ emanating from $\varnothing$, let f map $\rho \cap FB$ (a single point) onto $\rho \cap FC$ (a single point), and for each $x \in FB$ let f map the segment $[\varnothing, x]$ linearly onto $[\varnothing, fx]$. Then f is the "radial projection" of B onto C. In terms of the gauges $b = g_b$ and $c = g_c$, f can be defined as follows: $f\varnothing = \varnothing$ and $fx = (bx/cx)x$ for $\varnothing \neq x \in B$.

For each gauge g on L and point x of L, let $g^*x = \max [gx, g(-x)]$. It is not hard to verify that g^* is a gauge satisfying

$$g^*(tx) = |t| (g^*x). \tag{1'}$$

(The gauges g^* are characterized by having $C_{g^*} = -C_g$.). From (1') and (2) follow

$$g^*(rx + sy) \leqq |r| (g^*x) + |s| (g^*y) \tag{A}$$

and

$$|gx - gy| \leqq g^*(x - y). \tag{B}$$

We can now prove the

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THEOREM. *Suppose B and C are bounded closed convex bodies in a normed linear space E, each including $\varnothing$ in its interior. Then the radial projection of B onto C is Lipschitzian.*

Proof: Let $b=g_B$ and $c=g_C$. Let a be the norm of E, so that $a=a^*=g_U$, where U is the unit cell of E. Since B and C are both bounded and each includes $\varnothing$ in its interior there is a number $r>1$ such that $r^{-1}U\subset B\cap C\subset B\cup C\subset rU$, and then (D) $r^{-1}a\leqq b,c\leqq ra$. To prove the theorem, it suffices to find a number $N<\infty$ such that

$$a\left[\frac{bu}{cu}u-\frac{bv}{cv}v\right]\leqq Na(u-v)$$

whenever $u, v\in B$ and $u\neq\varnothing\neq v$. Consider an arbitrary such pair of points with, say, $cv\leqq cu$. Let $t=(cu)^{-1}$, $x=tu$, and $y=tv$. Then $cy\leqq cx=1$ and $y, x\in C$. We have

$$\begin{aligned}
ta\left[\frac{bu}{cu}u-\frac{bv}{cv}v\right] &= a^*\left[\frac{bx}{cx}(x-y)+\left(\frac{bx}{cx}-\frac{by}{cy}\right)y\right]\\
&\leqq \frac{bx}{cx}a^*(x-y)+\left|(bx-by)+\frac{by}{cy}(cy-cx)\right|a^*y\\
&\leqq \frac{bx}{cx}a(x-y)+|bx-by|\,ay+\frac{by}{cy}|cy-cx|\,ay\\
&\leqq \left[\frac{bx}{cx}+ay\frac{b^*(x-y)}{a(x-y)}+ay\frac{by}{cy}\frac{c^*(x-y)}{a(x-y)}\right]a(x-y)\\
&\leqq Ma(x-y)=tMa(u-v),
\end{aligned}$$

where $M=(\sup_C b/c)+(\sup_C a)(\sup_{C-C} b^*/a)+(\sup_C a)(\sup_C b/c)(\sup_{C-C} c^*/a)$. (Here $C-C$ is the set $\{x-y:x, y\in C\}$.) In establishing these inequalities, (3) is used throughout. The first inequality uses (A) for $a=a^*$ and the fact that $cx=1$; the third uses (B) for b and c. From (D) it follows that $M<\infty$, completing the proof.

In closing, we note that the results of Vincze and Szüsz [3] have been extended by Haupt [2], and that some of Haupt's results can be obtained by the method used here.

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ON A THEOREM OF KIRZBRAUN AND VALENTINE

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THEOREM 1. *In the euclidean space E_n we are given a set of m spheres*

$$S_i: \quad |x - x_i| \leqq r_i, \qquad (i = 1, \cdots, m),$$

which have a point in common:

$$\bigcap_{i=1}^{m} S_i \neq 0.$$

If the spheres

$$S_i': \quad |x - x_i'| \leqq r_i, \qquad (i = 1, \cdots, m),$$

are such that

$$|x_i' - x_j'| \leqq |x_i - x_j|, \qquad (i, j = 1, \cdots, m), \tag{1}$$

then

$$\bigcap_{i=1}^{m} S_i' \neq 0. \tag{2}$$

In other words: If a set of m closed spheres in E_n have a common point then again they will have a common point if we displace their centers so as not to be further apart than they were before. This theorem is seen to be equivalent to the following:

THEOREM 2. *Let x_i, x_i' $(i=1, \cdots, m)$ be points in E_n satisfying* (1). *Let also the point p be given. Then there exists a point p' such that*

$$|p' - x_i'| \leqq |p - x_i|, \qquad (i = 1, \cdots, m). \tag{3}$$

Theorem 2 implies Theorem 1, for if $p \in \bigcap S_i$ then, by (3), $|p' - x_i'| \leqq |p - x_i| \leqq r_i$ or $p' \in \bigcap S_i'$, which proves (2). It is equally obvious that Theorem 1 implies Theorem 2. This result is due to Kirzbraun [1]. It was rediscovered by Valentine [3], [4]. Mickle [2] gave a new proof. Valentine uses in his proof a theorem of Knaster, Kuratowski and Mazurkiewicz. Mickle uses certain quadratic forms which were used by the present writer to characterize E_n metrically. The following is a blend of the proofs of Valentine and Mickle which avoids the use of the auxiliary tools just mentioned.

Proof of THEOREM 2: We lose no generality by assuming that $p \neq x_i$ $(i=1, \cdots, m)$, for if $p = x_j$, then $p' = x_j'$ will satisfy (3). Consider the function

$$f(x) = \max_i \frac{|x - x_i'|}{|p - x_i|},$$

which is defined and continuous throughout E_n. Since $f(x) \to +\infty$ as $|x| \to \infty$,

$f(x)$ attains its absolute minimum λ. Let $x=p'$ be such that

$$f(p') = \min_{x \in E_n} f(x) = \lambda, \tag{4}$$

and let us show that p' satisfies (3). If $\lambda=0$ then $p'=x_1'=\cdots=x_m'$ and (3) are trivially verified. Assume, therefore, that $\lambda>0$. Let

$$\frac{|p'-x_i'|}{|p-x_i|} \begin{cases} = \lambda & \text{if } i=1,\cdots,k \\ < \lambda & \text{if } i=k+1,\cdots,m, \end{cases} \qquad (1 \leqq k \leqq m), \tag{5}$$

as may always be achieved by a proper numbering of the points. Let $K(x_1', \cdots, x_k')$ denote the convex hull of the points $x_1', \cdots, x_k'$. We claim that

$$p' \in K(x_1', \cdots, x_k'). \tag{6}$$

Indeed, otherwise we could diminish all distances $|p'-x_i'|$, $(i=1,\cdots,k)$, by displacing p' so slightly that all m ratios (5) become less than λ. This, however, will contradict the definition (4) of λ. This result implies, in particular, that $k \geqq 2$ for $k=1$ would contradict our assumption that λ is positive.

In view of (5), the inequalities (3) will be established as soon as we can show that

$$\lambda \leqq 1. \tag{7}$$

In order to prove (7) let us assume that

$$\lambda > 1 \tag{8}$$

and see how we reach a contradiction. In terms of the vectors

$$R_i = x_i - p, \qquad R_i' = x_i' - p', \qquad (i = 1, \cdots, k),$$

(5) and (8) imply the inequalities

$$R_i'^2 > R_i^2, \qquad (i = 1, \cdots, k), \tag{9}$$

where R_i^2 denotes the scalar product of R_i by itself. Now (1) may be written as

$$(R_i' - R_j')^2 \leqq (R_i - R_j)^2, \qquad (i, j = 1, \cdots, k). \tag{10}$$

Expanding (10) and subtracting appropriate inequalities (9) we have

$$R_i' \cdot R_j' > R_i \cdot R_j, \qquad (i, j = 1, \cdots, k). \tag{11}$$

On the other hand (6) shows that p' is a centroid of the points $x_1', \cdots, x_k'$. This means: There exist $c_i \geqq 0$ with $\sum c_i = 1$ such that

$$p' = \sum_{i=1}^{k} c_i x_i'.$$

But then

$$0 = \sum c_i x_i' - p' = \sum c_i x_i' - \sum c_i p'$$

or

$$\sum_{i=1}^{k} c_i R_i' = 0, \qquad c_i \geqq 0, \ \sum c_i = 1. \tag{12}$$

A contradiction is now easily reached: on multiplying (11) by $c_i c_j$ and summing over i and j we obtain the inequality

$$\left(\sum_{1}^{k} c_i R_i' \right)^2 > \left(\sum_{1}^{k} c_i R_i \right)^2$$

which is obviously impossible for its left side vanishes by (12) while its right side is non-negative.

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3. F. A. Valentine, On the extension of a vector function so as to preserve a Lipschitz condition, Bull. Amer. Math. Soc., vol. 49, 1943, pp. 100–108.
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CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

A NOTE ON THE BASE OF NATURAL LOGARITHMS

ERNEST LEACH, Massachusetts Institute of Technology

A quick approach to the theory of natural logarithms starts from the defining formula

$$\ln x = \int_1^x t^{-1} dt. \tag{1}$$

The base e is then defined implicitly by:

$$\int_1^e t^{-1}dt = 1. \tag{2}$$

It is intuitively evident that if we vary the exponent in the integrand slightly, we shall have an approximation to e. That is, if we define E_α by:

$$\int_1^{E_\alpha} t^{-1+\alpha}dt = 1 \tag{3}$$

we should then have the relation

$$e = \lim_{\alpha \to 0} E_\alpha. \tag{4}$$

If $\alpha \neq 0$, we may solve for E_α quite easily, and find that

$$E_\alpha = (1 + \alpha)^{1/\alpha}.$$

Thus (4) is completely equivalent to the familiar relation

$$e = \lim_{\alpha \to 0} (1 + \alpha)^{1/\alpha}. \tag{4'}$$

We can use equation (3) to obtain a proof of the equivalent relations (4) and (4'). One method is to follow the classical proof of (4'), which is quite simple after it is known that E_α is a monotone function of α. But equation (3) gives this monotoneness as a trivial consequence of the monotoneness of the integrand in (3) as a function of α.

Another proof of (4) can be obtained by a simple continuity argument using (2) and (3). For, given $\epsilon > 0$, we have easily

$$\int_1^{e+\epsilon} t^{-1}dt > 1. \tag{5}$$

Then, by uniform continuity of $t^{-1+\alpha}$, we have for sufficiently small α,

$$\int_1^{e+\epsilon} t^{-1+\alpha}dt > 1, \tag{7}$$

from which we conclude that $E_\alpha < e+\epsilon$, for small α. In the same manner we prove that $E_\alpha > e-\epsilon$, for sufficiently small α, from which the relation (4) follows, since ϵ is arbitrary.

Although the elementary theory of the natural logarithm and the exponential function can be developed without recourse to (4'), this relation is useful in giving sidelights of the theory. As well-known examples, we remark that the series expansion for e^x can be derived from (4') and the binomial theorem, and that the Stirling approximation to $\sqrt[n]{n!}$ can be derived from (4') with the aid of the multiplicative analogue of Césaro summation.

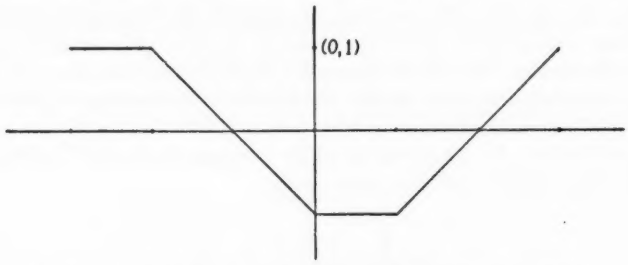
A KIND OF PROBLEM THAT EFFECTIVELY TESTS FAMILIARITY WITH FUNCTIONAL RELATIONS

K. O. MAY, Carleton College

The kind of problem described here probes the student's understanding of the functional concept and notation, tests his familiarity with the functions being studied, and is easy to run off on a duplicator and to correct.

Present the student with the graph of a rather simple function, perhaps piecewise constant or at least piecewise linear. Label this function $f(x)$ and then require the student to graph $-f(x)$, $f(-x)$, $|f(x)|$, $f(|x|)$, $e^{f(x)}$, $\sin f(x)$ or whatever functions involve the ideas whose understanding you wish to test. The correct graphs are often decorative and always easily recognized. The student's deviations indicate clearly in many cases the nature of his difficulties. While the resulting graphs are easy to check, it is not hard to construct examples that require considerable thought and challenge the best students. On the other hand, the simpler examples, such as $-f(x)$ or the inverse function of $f(x)$, appeal to the manipulative students. Obvious modifications, such as the use of two functions, will occur to any teacher.

As an example suppose that $y=f(x)$ has the graph:



Sketch: $y=|f(x)|$, $y=f(|x|)$, $y=-f(x)$, $y=e^{f(x)}$, $y=\text{Arc sin } f(x)$.

SOME REMARKS ON CENTROIDS

R. B. DEAL and W. N. HUFF, The University of Oklahoma

Recently it was pointed out by two students in our integral calculus classes that the ordinate of the centroid of area of one arch of the cycloid and that of the centroid of the volume of revolution of the arch about the Y-axis were equal. The question was raised as to the reason for this and the present note concerns this problem.

Let us consider an area, A, in the first quadrant, bounded by the curves $y=f(x)$ above and $y=g(x)$ below, the curves intersecting at (a, b) and (c, d). Let $(\bar{x}, \bar{y})$ be the centroid of area A, $(\bar{x}_1, 0)$ the centroid of V_1, the volume of rotation about the x-axis, and $(0, \bar{y}_2)$ the centroid of V_2, the volume of rotation about the y-axis.

Then

$$A\bar{y} = \frac{1}{2}\int_a^c (f^2 - g^2)dx = \frac{V_1}{2\pi},$$

$$V_2\bar{y}_2 = \pi\int_a^c (f^2 - g^2)xdx.$$

Since by the theorem of Pappus $V_2 = 2\pi\bar{x}A$, it follows that

$$2\pi\bar{x}A\bar{y}_2 = \pi\int_a^c (f^2 - g^2)xdx = \bar{x}_1V_1 = 2\pi A\bar{x}_1\bar{y}.$$

Hence $\bar{x}\bar{y}_2 = \bar{x}_1\bar{y}$.

This can also be done by double integrals as follows:

$$2\pi A\bar{y}\bar{x}_1 = V_1\bar{x}_1 = \int_a^c\int_g^f 2\pi xydydx = V_2\bar{y}_2 = 2\pi A\bar{x}\bar{y}_2.$$

The above result then follows.

If the area in question possesses an axis of symmetry $x = k$, then $\bar{x}_1 = \bar{x} = k$ and $\bar{y}_2 = \bar{y}$. For the arch of the cycloid $k = \pi a$.

ON THE CONVERGENCE OF THE p-SERIES

R. M. FOSTER and M. S. KLAMKIN, Polytechnic Institute of Brooklyn

In quite a few textbooks, we find the statement that the p-series,

$$\sum_{n=1}^{\infty} n^{-p}, \tag{1}$$

converges for $p > 1$, and diverges for $p \leqq 1$. This is not quite correct. It should rather be stated, that the series converges for p(constant) > 1, and diverges for $p \leqq 1$. An example illustrating this point is given by the series (1) with $p = 1 + (1/n)$. This series diverges since

$$\lim_{n\to\infty} \frac{n^{1+(1/n)}}{n} = 1, \tag{2}$$

and $\sum_{n=1}^{\infty} n^{-1}$ diverges. This example was given as a problem in this MONTHLY (Nov. 1948, p. 584). One of the solutions gave the following generalization of this problem:

If $\lim_{n\to\infty}\phi(n) = 0$, then $\sum_{n=1}^{\infty} n^{-[1+\phi(n)]}$ diverges. This latter statement, however, is not correct. For, let r be constant and let $n^{\phi(n)} = \psi(n) = (\ln n)^r$. Then

$$\lim_{n\to\infty} \phi(n) = \lim_{n\to\infty} \frac{\ln \psi(n)}{\ln n} = 0, \tag{3}$$

but the series

$$\sum_{n=1}^{\infty} \frac{1}{n\psi(n)} = \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^r} \tag{4}$$

converges for r(constant) >1, and diverges for $r \leqq 1$. Similarly, other functions $\psi(n)$ can be chosen such that the series converges or diverges.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1086. *Proposed by T. A. Bickerstaff, University of Mississippi*

"That was a good lunch; now for a good cigar and then I must catch the one o'clock train. Let's see—my watch says exactly nine o'clock but that can't be right. It's still running and well wound. Now I remember I wound it and set it just this morning by the radio. Maybe I carelessly set the hands in reverse position. If so, exactly what time is it?"

E 1087. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Let a be an arbitrary positive integer and let

$$S_m = \sum_{j=1}^{a} j^m.$$

We have

$$a(a+1)^2 = S_1 + 3S_2, \qquad a^2(a+1)^3 = S_2 + 2S_3 + 5S_4,$$
$$a^3(a+1)^4 = S_3 + 5S_4 + 3S_5 + 7S_6, \cdots .$$

Determine coefficients c_k such that

$$a^n(a+1)^{n+1} = \sum_{k=n}^{2n} c_k S_k.$$

E 1088. *Proposed by M. J. Pascual, Siena College*

A and B agree to play n games and he who goes first has a chance of $a/(a+b)$ of winning that game. If the winner of any game goes first in the following game, then find (1) the probability of A winning the nth game if he goes first in the opening game, (2) the expected value of A's winnings in n games if each game is worth d dollars.

E 1089. *Proposed by W. R. Utz, University of Missouri*

Show that if $f(x)$ is a real function bounded below on $I=[a, b]$ and $g(x)$ is real and either (i) monotone increasing on I or (ii) continuous with $g(b)>g(x)$ for $a\leqq x\leqq b$, then given $\epsilon>0$ there is a constant $\lambda(\epsilon)>0$ such that $f(x)-\lambda g(x)$ cannot attain $\inf_{x\in I}[f(x)-\lambda g(x)]$ on $[a, b-\epsilon]$. (The special case $g(x)=x$ is used by E. Baiada, *Ann. Scuola Norm. Super. Pisa*, vol. 15 (1950), p. 111.)

E 1090. *Proposed by B. M. Stewart, Michigan State College*

From one vertex of a triangle lines are to be drawn dividing the triangle into a set S of n triangles having equal inscribed circles.

(1) Show that in general the set S may be constructed by ruler and compass if and only if $n=2^s$.

(2) Show that the $n-k+1$ triangles formed by taking sets of k adjacent triangles of the set S have equal inscribed circles ($k=2, 3, \cdots, n-1$).

(3) Find a neat construction when $n=2$. (*Note.* From (2) it follows that repeated application of (3) will solve the problem when $n=2^s$.)

SOLUTIONS

The Spider and the Fly

E 1056 [1953, 188]. *Proposed by W. R. Ransom, Tufts College*

A rectangular room has a spider one foot down from the ceiling at the middle of one end, and a fly one foot up from the floor at the middle of the other end. There are three paths by which the spider can crawl to the fly, and which become straight lines when the sides of the room are properly developed. What dimensions of the room will make these three paths equal in length?

Editorial Note. There seems to be no easily located treatment of the spider and fly problem which considers the maximum number n of possible geodesic routes from the spider to the fly, and the number d of these routes which have distinct lengths, in terms of the length l, width w, and height h of the room, and the positions of the spider and fly on the end walls. What is the smallest possible value for n, and, for given n, what are the greatest and least possible values for d? What is the shortest route? Kasner and Newman in *Mathematics and the Imagination*, p. 182, indicate that $n\geqq 4$. Kraitchik in *Mathematical Recreations*, pp. 17–21, indicates that $n\geqq 8$, and shows that for $l:w:h=390:240:260$ we may have all eight routes equal in length. Boblétt constructed a model

with $l:w:h=5:2:2$ on which 16 routes are indicated, these routes being equal in symmetrical pairs. It is easy to show that there exist rooms (very flat ones) having n as large as may be desired.

Contributions by A. P. Boblétt, Julian Braun, Mary Dean Clement and J. D. Haggard (jointly), J. R. Hatcher and C. E. Jones (jointly), Frank Herlihy, R. E. Jackson, M. S. Klamkin, F. A. Lee, Jr., C. S. Ogilvy, Bart Park, M. J. Pascual, and the proposer.

A Series Involving Secants

E 1057 [1953, 188]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Find the sum of the first n terms of the series

$$\sec\theta + (\sec\theta \sec 2\theta)/2 + (\sec\theta \sec 2\theta \sec 4\theta)/4 + \cdots .$$

Solution by J. R. Hatcher, Fisk University. Using the identities $\sec\alpha = 2\sin\alpha/\sin 2\alpha$ and $\csc\alpha = \cot\alpha/2 - \cot\alpha$, we conclude that the sum $S_n(\theta)$ of the first n terms is

$$\begin{aligned} S_n(\theta) &= 2\sin\theta(\csc 2\theta + \csc 4\theta + \cdots + \csc 2^n\theta) \\ &= 2\sin\theta(\cot\theta - \cot 2^n\theta) \\ &= [2\sin(2^n - 1)\theta]/[\sin 2^n\theta]. \end{aligned}$$

Also solved by H. M. Feldman, N. J. Fine, H. W. Gould, Eugene Malek, Robert Muguercia, S. Parameswaran, L. A. Ringenberg, Francis Sévier, F. Underwood, Chih-yi Wang, J. V. Whittaker, and the proposer.

Underwood pointed out that the problem is essentially Ex. 10, p. 125 of *Plane Trigonometry, Part II*, by S. L. Loney (1908).

Two Properties of the Divisors of an Integer

E 1058 [1953, 188]. *Proposed by M. Perisastri, M. R. College, Vizianagram, India*

If $d_1, d_2, \cdots, d_k$ are the divisors of n, show that

$$(1) \qquad (d_1 d_2 \cdots d_k)^2 = n^k,$$

$$(2) \qquad d_1^{d_1} d_2^{d_2} \cdots d_k^{d_k} = n^{d_1+d_2+\cdots+d_k} e^{-nrk}, \qquad 0 \leqq r \leqq (\log 3)/3.$$

Solution by W. E. Briggs, University of Colorado. If $d_1 = 1, d_2, \cdots, d_k = n$ is a complete set of divisors of n, then $n/d_1, n/d_2, \cdots, n/d_k$ is another complete set. Therefore

$$d_1 d_2 \cdots d_k = (n/d_1)(n/d_2)\cdots(n/d_k) = n^k/d_1 d_2 \cdots d_k,$$

whence

$$(d_1 d_2 \cdots d_k)^2 = n^k.$$

Next let $d_i = n/d_{k-i+1}$, $i = 1, 2, \cdots, k$. Then

$$d_1^{d_1} d_2^{d_2} \cdots d_k^{d_k} = (n/d_k)^{d_1}(n/d_{k-1})^{d_2} \cdots (n/d_1)^{d_k} = n^{d_1+d_2+\cdots+d_k}/d_1^{d_k} d_2^{d_{k-1}} \cdots d_k^{d_1},$$

and

$$1/d_1^{d_k} \cdots d_k^{d_1} = 1/d_1^{n/d_1} \cdots d_k^{n/d_k} = (d_1^{1/d_1} \cdots d_k^{1/d_k})^{-n}.$$

But it is well known that for all positive integers m, $1 \leqq m^{1/m} \leqq 3^{1/3}$. Therefore

$$1 \leqq d_1^{1/d_1} \cdots d_k^{1/d_k} \leqq 3^{k/3} = e^{\log 3^{k/3}} = e^{k(\log 3)/3}$$

and result (2) follows.

Also solved by N. J. Fine, Vern Hoggatt, M. S. Klamkin, A. E. Livingston, Azriel Rosenfeld, J. V. Whittaker, and the proposer.

A Polygon Theorem

E 1059 [1953, 188]. *Proposed by Chih-yi Wang, Hampton Institute*

Let a circle and an inscribed closed polygon of n sides be given. Show that the product of the distances of a point on the circumference of the circle from the sides of the polygon is equal to the product of the distances of the same point from the sides of the tangential polygon (*i.e.*, the polygon formed by the tangents to the circle at the vertices) of the given polygon.

I. *Solution by L. J. Lander, University of California.* Take the circle as the unit circle, and the point on the circumference as (1, 0). Let $V_1, V_2, \cdots, V_n$ be the vertices of the inscribed polygon, and let OV_k make an angle of $2\theta_k$ with the x-axis (where O is the origin). If d_k is the distance of (1, 0) from the side V_kV_{k+1}, and D_k is its distance from the side of the tangential polygon passing through V_k, we have

$$d_k = \cos(\theta_{k+1} - \theta_k) - \cos(\theta_{k+1} + \theta_k) = 2 \sin\theta_k \sin\theta_{k+1},$$

$$D_k = 1 - \cos 2\theta_k = 2 \sin^2 \theta_k.$$

It is now clear that $\Pi d_k = \Pi D_k$.

II. *Solution by C. W. Trigg, Los Angeles City College.* This is a generalization of 3726 [1937, 58] wherein is proven the lemma: the perpendicular from a point on a circle to a chord is the mean proportional between the perpendiculars to the tangents at the extremities of the chord. Let x_i, y_i denote the lengths of the perpendiculars from the point to the chord and to a tangent, $(i = 1, 2, \cdots, n)$. Then $x_i = \sqrt{y_i y_{i+1}}$, $x_{i+1} = \sqrt{y_{i+1} y_{i+2}}$, $\cdots$. It follows immediately that $\Pi x_i = \Pi y_i$.

Also solved by W. B. Carver, Russell Godard and Vern Hoggatt (jointly), J. R. Hatcher, C. S. Ogilvy, S. Parameswaran, Michael Skaliskyj, Sister M. Stephenie, Roscoe Woods, and the proposer.

Carver and Sister Stephanie used complex coordinates. Carver remarked that a set of n vertices leads to $(n-1)!/2$ different inscribed n-gons, all having

the same tangential polygon. When $n=4$ there are three inscribed polygons. Applying the result of the problem to these three inscribed polygons one can readily show that *in a complete cyclic quadrangle the product of the distances of any point on the circumcircle from any pair of opposite sides of the quadrangle is equal to the product of the distances of the point from any other pair of opposite sides of the quadrangle.*

Woods pointed out that the problem is stated as a theorem on p. 72 of Johnson's *Modern Geometry.*

The feet of the perpendiculars dropped on the sides of the tangential polygon lie on a cardioid (see, *e.g.*, Yates' *Curves and Their Properties*, 4 (c), p. 5).

Multinomial Coefficients

E 1060 [1953, 188]. *Proposed by H. K. Crowder, Case Institute of Technology*

Let D be a determinant of order n whose ith row, $i=1, \cdots, n$, is

$$a_i, a_{i-1}, \cdots, a_1, 1, 0, \cdots, 0.$$

Show that

$$D = (-1)^n \sum \frac{(-1)(-2)\cdots(-\sum \alpha_i)}{\alpha_1!\alpha_2!\cdots\alpha_n!} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n},$$

where the sum is taken over all non-negative integral solutions of

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n.$$

Solution by the Proposer. The sum is the coefficient, for x sufficiently small, of x^n in the expansion of

$$(1 + a_1x + a_2x^2 + \cdots + a_nx^n)^{-1}$$

(see, *e.g.*, Chrystal, *Textbook of Algebra, Part II*, 2nd ed. (1900), p. 214). Let

$$(1 + a_1x + a_2x^2 + \cdots + a_nx^n)^{-1} = 1 + b_1x + b_2x^2 + \cdots.$$

Then

$$(1 + a_1x + a_2x^2 + \cdots + a_nx^n)(1 + b_1x + b_2x^2 + \cdots) = 1,$$

and collecting coefficients we have

$$\begin{aligned} &a_1 + b_1 = 0, \\ &a_2 + a_1b_1 + b_2 = 0, \\ &a_3 + a_2b_1 + a_1b_2 + b_3 = 0, \\ &\cdots\cdots\cdots\cdots\cdots \\ &a_n + a_{n-1}b_1 + a_{n-2}b_2 + \cdots + b_n = 0. \end{aligned}$$

Solving this system of linear equations in the b_i for b_n we easily find $b_n = (-1)^n D$.

A sidelight is that the number of solutions of

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$$

is the number of partitions $P(n)$ of the integer n, so that the number of terms in the expansion of the determinant D is $P(n)$.

Also solved by N. J. Fine and F. D. Parker.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4558. *Proposed by V. F. Ivanoff, San Francisco*

If an octagon, $l_1l_2 \cdots l_8$ is inscribed in a conic, then the eight points of intersection of the sides l_i and $l_j (j \equiv i+3, \text{ mod } 8)$ lie on another conic.

4559. *Proposed by Harry Pollard, Institute for Advanced Study, and Paul Erdös, American University*

Let $0 < a_1 \leqq a_2 \leqq \cdots$ and put

$$F(y) = \prod_{k=1}^{\infty} \left(1 + \frac{y^2}{a_k^2}\right).$$

The necessary and sufficient condition for the convergence of

$$\int_1^{\infty} \frac{\log F(y)}{y^2}\, dy$$

is that $\sum_{k=1}^{\infty} 1/a_k$ converges.

4560. *Proposed by L. J. Mordell, St. John's College, Cambridge, England*

Prove that there are exactly $(\lambda+1)^{n+1} - \lambda^{n+1}$ sets of n integers, $x_1, x_2, \cdots, x_n$, which satisfy the inequalities

$$|x_r| \leqq \lambda, \qquad |x_r - x_s| \leqq \lambda, \qquad (r, s = 1, 2, \cdots, n),$$

where λ is a positive integer.

4561. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

Prove

$$S_{2,p}+\sum_{n=1}^{r}\left[n^2\binom{n+p}{p}\right]^{-1}=S_{2,r}+\sum_{n=1}^{p}\left[n^2\binom{n+r}{r}\right]^{-1}$$

where $S_{2,p}=\sum_{n=1}^{p}1/n^2$.

4562. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC, let P be a point having normal coördinates (x, y, z) and consider the points A', B', C' with normal coördinates $(-x/2, y, z)$, $(x, -y/2, z)$, $(x, y, -z/2)$. (1) The points A, B, C, A', B', C' lie on one conic S, and there is a conic with respect to which the triangles ABC and $A'B'C'$ are self polar. (2) If AP, BP, CP cut BC, CA, AB in A_1, B_1, C_1 and if $A'P$, $B'P$, $C'P$ cut $B'C'$, $C'A'$, $A'B'$ in A_1', B_1', C_1', the triangles ABC, $A'B'C'$ are circumscribed about a conic Σ, the points of tangency being A_1, B_1, C_1 and A_1', B_1', C_1'. (3) The conics S and Σ have double contact along the common polar of P with respect to these conics.

SOLUTIONS

A Set of Irrational Numbers

4494 [1952, 412]. *Proposed by D. J. Newman, Harvard University*

Let $a_1<a_2<\cdots$ be a sequence of positive integers, and let ξ be the unending decimal fraction formed by juxtaposing the a's (e.g., if the a's are the primes, ξ would be $0.23571113\cdots$). Prove that if $\sum 1/a_n=\infty$, then ξ is irrational.

Solution by Robert Breusch, Amherst College. Assume ξ is rational, with a period $b_1b_2\cdots b_q$, $0\leqq b_j<10$. From a certain index n_0 on, the a_n will be formed from the b's; if $q+1$ of them should contain the same number of digits, then the last number of this sequence would necessarily be equal to the first one, contrary to the assumption that the a_n increase.

Thus

$$\sum_{n=n_0}^{\infty}\frac{1}{a_n}<q\cdot\sum_{n=0}^{\infty}\frac{1}{10^n}<11q.$$

But this is contrary to the divergence of $\sum 1/a_n$. Therefore ξ must be irrational.

Discussion by Fritz Herzog, Michigan State College. The above result should be considered in connection with the following theorem of Copeland and Erdös: *If $a_1, a_2, \cdots$ is an increasing sequence of integers such that for every $\theta<1$ the number of a's up to N exceeds N^θ provided N is sufficiently large, then the infinite decimal $\xi=0.a_1a_2a_3\cdots$ is normal with respect to the base in which these in-*

*tegers are expressed.** A number is normal provided each of the digits 0, 1, 2, $\cdots$, 9 occurs with a limiting relative frequency of 1/10 and each of the 10^k sequences of k digits occurs with the frequency 10^{-k}.

To the question as to whether the numbers of problem 4494 are necessarily normal a negative answer is provided by the following example. We select the numbers a_n in groups which will be referred to alternately as "good" groups and "bad" groups. The first good group is to consist of the numbers 1, 2, $\cdots$, 9. These are followed by the first bad group, viz. the numbers 99, 999, $\cdots$, $10^{m_1}-1$, where m_1 is so large that the relative frequency of the digit 9 among the numbers selected so far exceeds $\frac{1}{2}$. Next we take for the second good group all numbers from 10^{m_1} to $10^{m_1+1}-1$, *i.e.*, all numbers with m_1+1 digits. For the second bad group we take the numbers $10^{\mu}-1$, $\mu=m_1+2, m_1+3, \cdots, m_2$, where again m_2 is so large that the relative frequency of the digit 9 among all the numbers selected up to this point is greater than $\frac{1}{2}$. Continuing in this manner we obtain a sequence $\{a_n\}$ and a number ξ. The latter is obviously not normal since the relative frequency of the digit 9 is greater than $\frac{1}{2}$ at the end of every bad group. On the other hand $\sum_n 1/a_n=\infty$, since the sum of the reciprocals of the a_n of any good group is greater than log 10.

Also solved by N. J. Fine, Harry Furstenberg, Fritz Herzog, M. S. Klamkin, Norman Miller, Leo Moser, M. Pearl, L. L. Pennisi, Louis Weiner, and the Proposer.

A Restricted Mean Value Theorem for Complex Variables

4495 [1952, 412]. *Proposed by M. S. Wertheim, Ithaca, New York*

Find all regular functions of a complex variable for which the mean value theorem holds for all a, b, in the form

$$\frac{F(b)-F(a)}{b-a}=F'(c)$$

where c is on the straight line segment ab.

Solution by N. J. Fine, University of Pennsylvania. Let a be a point such that $F''(a)\neq 0$ (leaving aside the trivial case where F is linear). We may assume that $a=0$ and $F(a)=0$. Since $F''(0)\neq 0$, the equation

$$F'(w)=\frac{F(z)}{z}$$

defines w as an analytic function of z in the neighborhood of the origin, with $w(0)=0$. By assumption, however, $w(z)=z\cdot\theta(z)$ where $\theta(z)$ assumes only real values. So $\theta(z)$ is constant, whence F satisfies

$$F'(\theta z)=\frac{F(z)}{z}\cdot$$

* Bull. Amer. Math. Society, v. 52 (1946), pp. 857–860.

It is now easy to see, by equating coefficients in the power series expansions of both sides, that $\theta=\frac{1}{2}$ and $F^{(n)}(0)=0$ for $n>2$. Thus $F(z)$ is quadratic.

Also solved by O. E. Stanaitis and the Proposer.

Entire Functions with Prescribed Complex Roots

4498 [1952, 469]. *Proposed by D. J. Newman, Harvard University, and H. S. Shapiro, Chatham, New Jersey*

Prove that every complex number is a zero of some entire function whose power series has rational coefficients.

Solution by Fritz Herzog, Michigan State College. Let $A(z)=\sum a_n z^n$ be any entire function. If $A^*(z)=\sum \overline{a_n} z^n$ then $A(z)\cdot A^*(z)$ is an entire function whose power series has real coefficients.

Let $B(z)=\sum b_n z^n$ be any entire function with real b_n. Then there exists an entire function $C(z)=\sum c_n z^n$ (with $C(z)\not\equiv 0$), such that the power series of $B(z)\cdot C(z)$ has rational coefficients. Indeed, if b_n is the first non-vanishing coefficient among the b's, we merely have to choose $c_0, c_1, c_2, \cdots$, successively in such a way that $-1/m!<c_m<1/m!$ and that the number

$$b_h c_m + b_{h+1}c_{m-1} + \cdots + b_{h+m}c_0$$

becomes rational. (To ensure $C(z)\not\equiv 0$, choose $c_0\neq 0$.)

The statement of the proposed problem is proved by application of the above procedure to $A(z)=\omega-z$, where ω is the given complex number. Moreover, by the use of Weierstrass' Product Theorem, we can extend that statement from a single point ω to any given point-set in the complex plane which has no finite limit-point.

Also solved by P. J. Cohen, A. M. Gleason, Melvin Henriksen, V. Ganapathy Iyer, Joseph Lehner, L. L. Pennisi, and the Proposers.

Editorial Note. As pointed out by Henriksen, this is a known result. See A. Hurwitz (*Acta Mathematica*, 14 (1890–91), pp. 211–215), O. Helmer (*Duke Math. Journal*, 6 (1940), pp. 345–356), and J. Lehner (*Journ. London Math. Society*, 25 (1950), pp. 279–282).

Probability that a Congruence Has No Solution

4499 [1952, 469]. *Proposed by P. G. Federbush, Student, Massachusetts Institute of Technology*

The coefficients of the congruence

$$a_0x^n + a_1x^{n-1} + \cdots + a_n \equiv 0 \pmod{p}$$

are subject to the restriction $0\leqq a_j\leqq p-1$, $a_0\neq 0$, but otherwise are taken at random. p is a prime. What is the probability that the congruence has no solution?

Solution by Leonard Carlitz, Duke University. Consider the number $\phi_n(M)$ of primary polynomials

$$x^n + a_1x^{n-1} + \cdots + a_n \pmod{p}$$

of degree n, prime to a given polynomial $M(x) \pmod{p}$. Then by the argument used in elementary number-theory to evaluate the Euler ϕ-function we can show that

$$\phi_n(M) = p^n \prod_{P \mid M} \left(1 - \frac{1}{|P|}\right) \qquad (|P| = p^{\deg P}), \tag{1}$$

where the product is over all irreducible P dividing M, and it is understood that we retain only those terms of the expanded product which are of non-negative degree in p. (For a different proof of (1), see *American Journal of Mathematics*, vol. 54 (1932), p. 44.)

Now for the problem in hand we take $M = x^p - x \equiv x(x-1)\cdots(x-p+1) \pmod{p}$. Hence (1) becomes

$$\phi_n(x^p - x) = p^n\left(1 - \frac{1}{p}\right)^p,$$

with the same understanding as before about which terms to retain. Since the total number of primary polynomials of degree $n \pmod{p}$ is p^n, we see that the probability that the stated congruence have no solution is given by

$$p^{-p}(p-1)^p \qquad (n \geqq p)$$

$$\sum_{r=0}^{n} (-1)^r \binom{p}{r} p^{-r} \qquad (n < p).$$

Also solved by A. M. Gleason and the Proposer.

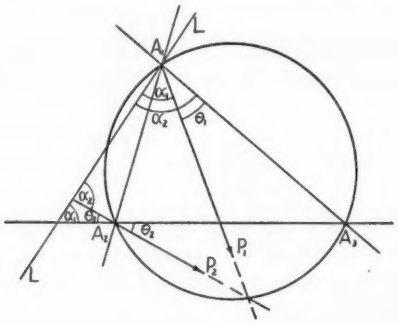
Editorial Note. Fritz Herzog refers to a paper by K. Zsigmondy, "Ueber die Anzahl derjenigen ganzzahligen Functionen n^{ten} Grades von x, welche in Bezug auf einen gegebenen Primzahlmodul eine vorgeschriebene Anzahl von Wurzeln besitzen," *Sitzungsber. Ak. Wiss. Wien, Math.-naturw. Kl.*, vol. 103 (1894), pp. 135–144. (See also L. E. Dickson, *History of the Theory of Numbers*, vol. I, p. 230.) The present problem is a special case of the one considered by Zsigmondy and the result may be obtained directly from his.

Concurrent Lines

4500 [1952, 469]. *Proposed by J. R. Musselman, Western Reserve University*

Given three points $A_i (i = 1, 2, 3)$ and a line L cut by the lines A_jA_k making angles α_i in the positive sense. Show that the lines drawn through A_i making angles $\pi - \alpha_i$, in the positive sense, with L are concurrent at a point P on the circumcircle of $A_1A_2A_3$. Further, if the altitudes with A_i make angles β_i with L

then the lines through A_i making angles $\pi-\beta_i$ with L are concurrent on the circumcircle of $A_1A_2A_3$ at a point Q diametrically opposite to P.



Solution by W. J. Robinson, Centre College, Kentucky. The line L can be replaced by any line parallel to L without changing the problem. Hence, take L through A_i so that it remains outside the triangle $A_1A_2A_3$, as in the drawing (where A_i is taken as A_1). The hypothesis of the problem is satisfied if the pairs of angles labeled α_1 and α_2 are equal. Let the lines drawn through A_1 and A_2 meet the circumcircle again in P_1 and P_2. Continuity and symmetry now complete the proof if we can show that A_1P_1 and A_2P_2 intersect on the circle.

But the angles labeled θ_1 and θ_2 are equal since each is $\alpha_2-\alpha_1$. Also θ_1 and θ_2 are measured respectively by half arc P_1A_3 and half arc P_2A_3. Hence the two arcs are equal and $P_1=P_2=P$.

Since β_i is the complement of α_i, the α-line through A_i is perpendicular to the β-line. But such lines meet at Q, the point diametrically opposite to P, since all inscribed angles are in semicircles.

Also solved by A. P. Boblétt, A. M. Gleason, Joseph Langr, and S. V. Venkataramana Rao.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of the Association.

Introduction to Measure and Integration. By M. E. Munroe. Cambridge, Mass., Addison-Wesley Publishing Co., Inc., 1953. 10+310 pages, $7.50.

This book has been written primarily as a textbook for a first course in measure and integration theory. It presupposes that the reader has a basic

knowledge of real function theory and in particular, that he is familiar with "ϵ, δ" method of proof. The author's approach is from the postulational point of view with emphasis given to a number of the important interpretations of the theory.

The first chapter is a review of that material of basic mathematics and real function theory which will be used in later sections, for example, algebra of sets, cardinal numbers, metric spaces, limits, and function spaces. Chapter II is concerned with the theory of additive set functions and measures. General procedures are given for constructing outer measures and metric outer measures.

Chapter III deals primarily with specific examples of measures, *i.e.*, Lebesgue-Stieltjes, Hausdorff, and Haar. It concludes with a discussion of nonmeasurable sets. Chapter IV opens with a discussion of measurable functions, and their basic properties. Approximation theorems are then developed. It concludes with a short section on stochastic variables. In Chapter V the integral is defined *via* simple functions and its basic properties developed. Absolute continuity, Fubini's theory, and the expectation of a stochastic variable are treated in the latter sections. Chapter VI deals with convergence theorems, L^p spaces, linear normed spaces, the mean ergodic theorem, and orthogonal expansions. Chapter VII is concerned with differentiation. It contains sections on Vitali coverings, differentiation of additive set functions, the Lebesgue decomposition, metric density and approximate continuity, and differentiation with respect to nets.

The format and typography of the book are pleasing. The author's choice of terminology and notation is modern. Pertinent references are listed at the end of each chapter. In addition to the usual index, an index of symbols and an index of postulates are provided.

The author's style is clear and concise. Motivation is provided and possible misunderstandings pointed out. The book was written with flexibility in mind. A number of the sections mentioned above are starred so that they may be taken up or omitted depending upon an instructor's conception of what topics should constitute a course in measure and integration. Exercises are abundant and play an important role. Besides being used to familiarize the reader with the terminology and to embellish the text material, they are used extensively to emphasize the importance of counter examples and to develop different approaches to the theory. It should be pointed out the topology used is of a metric nature and that in general the ultimate in generality has been avoided when it appears to unduly complicate the discussion.

This is an excellent textbook and in addition should prove to be a good reference for the non-specialist in integration.

E. H. CRISLER
Oberlin College

Introduction to the Foundations of Mathematics. By R. L. Wilder. New York, John Wiley and Sons, Inc., 1952. xiv+305 pages. $5.75.

Professor Wilder's course on the foundations of mathematics has for over twenty years been a source of intellectual stimulation for students from many departments of the University of Michigan. The publication of his book now affords others an opportunity to benefit from his experience as a student and teacher in this field. Although the nature of the subject requires of the reader an ability to handle abstract concepts, he will need very little in the way of mathematical background in order to appreciate the author's exposition.

The book is divided into two parts, the first, comprising nearly two thirds of the text, on fundamental concepts and the second on foundations. Part I begins with a discussion (two chapters) on the axiomatic method. This is illustrated by the development of a set of postulates for a portion of Euclidean geometry and of some of their consequences. Here, as throughout, historical comments and numerous problems add much to the value of the text. The next three chapters deal with set theory, including such topics as the Russell paradox, infinite sets, the Choice Axiom, and cardinal and ordinal numbers. A chapter on the real number system, and one on groups and related algebraic structures complete this section. Part II, after brief discussions of the views of Kronecker, Cantor, Boole, Frege, Peano, and Poincaré, and an outline of Zermelo's axiomatisation of set theory, contains chapters on logicism, intuitionism, and formalism, and closes with one, entitled "The Cultural Setting of Mathematics," in which the author expresses his own convictions concerning the nature of mathematics. There is an eleven page bibliography and the book is well indexed.

The latter part of the book is particularly valuable since there is little available in English for the student who wishes to learn something of the subjects treated therein. This is especially true of intuitionism, and the author's chapter on this, most of which is devoted to the very complicated intuitionist set theory, furnishes an excellent introduction to the subject. The chapter on formalism includes proofs of Gödel's theorems on incompleteness and consistency of formal systems (omitting, of course, the proofs of recursiveness of various number-theoretic functions).* In the final chapter the author argues that the scope of mathematics varies enormously with the culture, thus rendering impossible a definition of "mathematics."

Professor Wilder's book is without question a real addition to the still quite scanty collection of texts on the foundations of mathematics. Many students will certainly find the reading of it a stimulating experience and, through the excellent system of bibliographical references, will be encouraged to continue further study of the subject.

H. E. VAUGHAN
University of Illinois

* The example given on p. 258 of translating a statement about formulas into one about their Gödel numbers is, unfortunately, incorrect. Consider, for instance, the Gödel numbers of $F_1(x)$ and $F_3(x)$.

Numerical Solutions of Differential Equations. By W. E. Milne. New York, John Wiley and Sons, Inc., 1953. 11+275 pp. $6.50.

The author has succeeded admirably in presenting some of the principal techniques available for the numerical solution of ordinary and partial differential equations. This is the most thorough, practical and comprehensive treatise which has appeared in English. By reading carefully about twenty pages suggested in the preface the practical computer can handle almost any "decent" problem in ordinary differential equations.

Part I includes seven chapters: (1) Introduction; (2) Elementary Numerical Solution; (3) Analytical Foundations; (4) Methods Based on Numerical Integration; (5) Methods of Runge-Kutta. Methods Based on Higher Derivatives; (6) Systems of Equations. Higher Order Equations; (7) Two-Point Boundary Conditions. Part II, Partial Equations, contains the headings: (8) Explicit Methods. Parabolic and Hyperbolic Equations; (9) Linear Equations and Matrices; (10) Implicit Methods. Elliptic Equations; (11) Characteristic Numbers. Three Appendixes, (A) Round-off errors; (B) Large-scale computing machines, (C) The Monte Carlo method, are followed by fourteen pages of Bibliography and two Indexes.

One of the important features of the book is the careful analysis of error, both step by step, and total. For instance in section 41 one formula is used to predict or integrate ahead and another to correct, thus providing a measure of the truncation error. Moreover certain formulas give a bound for the maximum accumulated error over a range.

Chapter 3 starts with Taylor's series and ends with numerous Quadrature Formulas in terms of Ordinates, Backward Differences, and Central Differences. Considerable use is made of difference equations and their solution. In chapter 6 the D, D^2 are replaced by central difference operators and Method XIII involves the solution of a truncated central difference equation and its eventual correction.

In Part II one treats only some simpler equations for which less general methods have been devised. These include either an approximate difference equation in two variables or a system of algebraic equations. Other topics treated are general boundary conditions, the point pattern, variable coefficients, curved boundaries, relaxation, iteration, accelerating convergence, orthogonality, and removal of singularities on the boundary.

There are fifty formal computations and many exercises. Answers to a number of these have been computed. Several typographical errors were found but on the whole the printing, figures and stencils are clear and the binding excellent.

C. C. Camp
University of Nebraska

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

NOTE ON MAGIC CIRCLES

G. E. RAYNOR, Lehigh University

In a recent paper* S. W. McInnis raised the following two questions: Is the number of perfect magic circles finite? If finite, can an example of each different set be illustrated?

That the answer to the first question is in the negative can be seen at once from the following two observations.

(1) Any magic square can be deformed into a perfect magic circle so that the columns of the square become the radii of the circle and the rows of the square become the rings of the circle.

(2) It is well known that magic squares of all orders, except of order two, exist.

As a matter of fact, the number of perfect magic circles with n radii is even greater than the number of magic squares of order n since in forming the circle as described above the diagonal property of the magic square is not needed. Thus a perfect magic circle can be formed from any semi-magic square.

A THEOREM FOR THE THEORY OF NUMBERS

ALBERTO CORAZAO, Escuela Militar de Chorrillos, Lima, Peru

If N is a composite integer whose positive factors are p and q, then N can be expressed as the sum of three addends, p, q and t, all of which are connected intimately to some arithmetical progressions. To this end, we state the following theorem.

THEOREM. *If p and q are two positive integers and if t is an integer representing the pth term of an arithmetical progression whose first term is -1 and whose common difference is $q-1$, then*

$$pq = p + q + t. \tag{1}$$

Proof: The formula for the pth term, t, of an arithmetical progression whose first term is -1 and whose common difference is $q-1$ is

$$t = -1 + (p-1)(q-1)$$

whence

$$t = -1 + pq - p - q + 1 = pq - p - q,$$

which shows the validity of (1).

* This MONTHLY, vol. 60, No. 5, 1953, pp. 347-351.

Now suppose we want to get the product of any two integers. We start by building up the array below. We write out a number of arithmetical progressions with -1 as their first term in each case, and having the common difference 0, 1, 2, 3, $\cdots$, respectively, in each succeeding progression, and then arrange them in such a way that their first, second, third, . . . , terms appear in vertical columns. If we number the rows of the array by numbers that run vertically

	1	2	3	4	5	6	7	8	9	·	p	·	n
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	·	·	·	-1
2	-1	0	1	2	3	4	5	6	7	·	·	·	$n-2$
3	-1	1	3	5	7	9	11	13	15	·	·	·	$2n-3$
4	-1	2	5	8	11	14	17	20	23	·	·	·	$3n-4$
5	-1	3	7	11	15	19	23	27	31	·	·	·	$4n-5$
6	-1	4	9	14	19	24	29	34	39	·	·	·	$5n-6$
7	-1	5	11	17	23	29	35	41	47	·	·	·	$6n-7$
8	-1	6	13	20	27	34	41	48	55	·	·	·	$7n-8$
9	-1	7	15	23	31	39	47	55	63	·	·	·	$8n-9$
·	·	·	·	·	·	·	·	·	·	·	·	·	·
q	-1	$q-2$	·	·	·	·	·	·	·	·	t_{qp}	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·
n	-1	·	·	·	·	·	·	·	·	·	·	·	·

from 1 to n, these numbers will keep a certain relation with the common difference; in fact, the number 1 is in front of the progression whose common difference is 0, the number 2 is in front of the progression whose common difference is 1, and the number q is in front of the progression whose common difference is $q-1$. The second factor of our desired product will be represented by the appropriate number in this column. In similar manner we number the vertical columns by numbers that run horizontally from 1 to n. The first factor p of the product will be represented by a number in this row. We shall denote the number in the qth row and pth column by t_{qp}. Since t is the pth term of the progression that starts with -1 and has the common difference $q-1$, we have, according to the theorem, and from the array,

$$pq = p + q + t_{qp}. \tag{2}$$

Take for example:

$$6 \times 8 = ?$$

Here $p=6$, $q=8$, and from the array $t_{8:6}=34$. Placing these values in (2), we have

$$6 \times 8 = 6 + 8 + 34 = 48.$$

By (2),

$$qp = q + p + t_{pq}.$$

Since $qp = pq$, it follows that

$$t_{qp} = t_{pq}.$$

But the values of t are based on two different progressions. To illustrate, consider

$$8 \times 5 = 8 + 5 + t.$$

Here t, from the array, comes from the progression

$$-1, 3, 7, 11, 15, 19, 23, 27, \cdots$$

and

$$8 \times 5 = 8 + 5 + 27 = 40.$$

On the other hand, in the product

$$5 \times 8 = 5 + 8 + t,$$

t, from the array, comes from the progression

$$-1, 6, 13, 20, 27, \cdots$$

and

$$5 \times 8 = 5 + 8 + 27 = 40.$$

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

College of St. Thomas announces: Professor F. C. Smith has been named Chairman of the Department of Mathematics; Dr. Kenneth McMillin has been appointed to an assistant professorship; Mr. Roy Dowling has been appointed to an instructorship.

At the University of South Carolina: Dr. Stephen Kulik of Claremont Men's

College has been appointed to an associate professorship; Dr. H. S. Collins of the University of Maryland has been appointed to an assistant professorship; Professor D. H. Clanton of Allen Military Academy and Mr. D. T. Walker, previously at the University of Georgia, have been appointed to instructorships.

Wayne University announces: Associate Professor Max Coral has been promoted to a professorship; Dr. Fred Brafman has been promoted to an assistant professorship; Professor A. F. Stevenson, formerly of the University of Toronto, has been appointed Associate Professor of Mathematics and Physics; Professor J. W. Baldwin, who retired in June, 1953, has been given the title of Professor Emeritus; Mr. Walter Hoffman of the University of Michigan and Mr. H. T. Slaby, University of Wisconsin, have been appointed to instructorships; Dr. S. T. C. Moy of the University of Illinois has been appointed to a part-time instructorship.

Mr. M. J. Aegerter, previously a graduate student at the University of Wisconsin, has accepted a position as a mathematician at the United States Naval Ordnance Test Station, China Lake, California.

Mr. J. T. Ahlin, who has been a research engineer with Douglas Aircraft Company, Santa Monica, California, is now Applied Science Representative with International Business Machines Corporation, Santa Monica.

Mr. J. E. Alman of Boston University has been appointed Director of the Office of Statistical and Research Services of the University.

Miss Florence R. Anderson, formerly a research laboratory analyst with Northrop Aircraft, Hawthorne, California, has accepted a position as a research engineer with North American Aviation, Los Angeles, California.

Assistant Professor H. M. Anderson of Gustavus Adolphus College has been promoted to an associate professorship.

Mr. J. M. Anderson, previously a student at the University of South Dakota, has been appointed to a graduate assistantship at Iowa State College.

Mr. D. L. Arenson, who has been Senior Research Engineer at the Cook Research Laboratory, Skokie, Illinois, is now Technical Director of the Aerophysics Section.

Assistant Professor Grace E. Bates of Mount Holyoke College has been promoted to an associate professorship.

Assistant Professor J. W. Beach of the University of New Mexico has been appointed to an associate professorship at Northern Illinois State Teachers College.

Assistant Professor Helen P. Beard of Newcomb College, Tulane University, is on leave of absence for the year 1953–54 and is at the Statistical Laboratory, University of California.

Assistant Instructor Imogene C. Beckemeyer of Southern Illinois University has been promoted to an instructorship.

Dr. V. N. Behrns of the University of Buffalo has accepted a position as Senior Operations Analyst with the Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

At the University of Dayton, Reverend William Bellmer, professor of mathematics and head of the Department of Mathematics, has been named Associate Dean of Science.

Mr. K. S. Bergman, formerly a field assistant with the United States Geological Survey, Spokane, Washington, is now Assistant Attorney General of the State of Washington.

Professor R. V. Blair of Vanderbilt University has been given the title of Professor Emeritus.

Assistant Professor G. M. Bloom of Miami University has been promoted to an associate professorship.

Mr. G. F. Bradfield has a position as Acting Educational Director at the Andersen Air Force Base, Guam.

Miss Mary P. Burkart, previously an instructor at Trinity College, Washington, D. C., has a position as a mathematician in the Navy Department, David Taylor Model Basin, Washington, D. C.

Assistant Professor H. N. Carter of the University of Tulsa has been promoted to an associate professorship.

Miss Jo Ann Cipolla, who has been a student at Elmira College, has accepted a position as a mathematician with E. I. duPont de Nemours and Company, Wilmington, Delaware.

Associate Professor Paul Civin of the University of Oregon is at the Institute for Advanced Study for the academic year 1953–54.

Miss Helen E. Core, formerly a research assistant at the University of Michigan, has been appointed Head of the Department of Mathematics of Northwestern Michigan College, Traverse City.

Mr. D. W. Crowe, previously a teaching fellow at the University of Michigan, has been appointed to a research assistantship.

Dr. R. B. Deal of the University of Oklahoma has been appointed to an assistant professorship at Oklahoma Agricultural and Mechanical College.

Assistant Professor S. P. Diliberto of the University of California has been appointed Assistant Dean of the College of Letters and Science.

Professor W. E. Edington, formerly head of the Department of Mathematics of DePauw University, has been given the title of Professor Emeritus.

Professor Howard Eves of Champlain College has been appointed to a professorship at Harpur College.

Mr. E. A. Fay, formerly a graduate student at the University of California, has a position as a statistician with the United States Naval Ordnance Test Station, China Lake, California.

Dr. J. F. Foster of the University of Arizona has been promoted to an assistant professorship.

Associate Professor J. E. Freund of Alfred University has been promoted to a professorship.

Mr. A. R. Friedenheit, who has been a stress analyst with North American

Aviation, Columbus, Ohio, has accepted a position as a mathematician with Douglas Aircraft Company, Santa Monica, California.

Mr. D. A. Gorsline of Equitable Life Assurance Society, Albany, New York, has been promoted to the position of District Manager.

Mr. J. S. Griffin, Jr., of Tulane University has been promoted to an instructorship.

Dr. W. N. Hallett is now Training Director of Goodwill Industries of Pittsburgh.

Associate Professor H. J. Hamilton of Pomona College has been promoted to a professorship.

Mr. C. V. Hannan, III, who has been a radio engineer with the Civil Aeronautics Administration, Oklahoma City, has been promoted to the position of Supervisory Electronics Engineer.

Mr. R. C. Haseltine has been promoted to the position of Research Engineer, Burroughs Adding Machine Company, Philadelphia, Pennsylvania.

Miss Joy S. Heller, previously a student at Brooklyn College, has been appointed to a graduate assistantship at the University of Oregon.

Mr. John Hilzman, who has been a student at the University of Rhode Island, has been appointed to a graduate assistantship at Oregon State College.

Assistant Professor Jessie M. Hoag of Southwestern Louisiana Institute has been promoted to an associate professorship.

Instructor S. B. Hobbs of the University of New Hampshire has accepted a position as an assistant project engineer with Sperry Gyroscope Company, Great Neck, New York.

Assistant Professor P. G. Hodge of the University of California at Los Angeles has been appointed to an associate professorship at the Polytechnic Institute of Brooklyn.

Mr. R. R. Hohl, formerly a graduate assistant at Lehigh University, has been appointed to a graduate assistantship at Albright College.

Mr. F. X. Holzhauer, formerly a teaching fellow at the University of Detroit, is now a research engineer with Ford Motor Company, Dearborn, Michigan.

Mr. R. E. Horton of Los Angeles City College has replaced Mr. C. W. Trigg, Los Angeles City College, as editor of the Problems and Questions Department of *Mathematics Magazine*.

Mr. David Horwitz has been promoted to the position of Associate Engineer, Armour Research Foundation, Chicago, Illinois.

Miss Ruth A. Huffman, who has been engaged as a statistician at Tinker Air Force Base, Oklahoma City, has accepted a position as an engineer with Chance Vought Aircraft, Dallas, Texas.

Mrs. Elizabeth M. Hutcheson, previously a student at the University of Cincinnati, is now with the Operations Research Group of Arthur D. Little, Incorporated, Cambridge, Massachusetts.

Mr. W. R. Hydeman, formerly with the Department of Defense, Washington, D. C., is engaged as a mathematician with the Engineering Research Associates, Division of Remington Rand, Arlington, Virginia.

Mr. R. E. Jackson, previously a student at the University of California, has a position as a mathematician with International Business Machines Corporation, Los Angeles, California.

Dr. H. G. Jacob, who has been an assistant at Yale University, has been appointed to an assistant professorship at Louisiana State University.

Dr. T. A. Jeeves of the University of California has been promoted to an assistant professorship.

Mr. Sidney Kaplan has left his position as Research Mathematician, Planning Research Branch, Office Comptroller of the Army, and has joined the staff of the Computing Systems Engineering Section, Victor Division, Radio Corporation of America, Camden, New Jersey.

Mr. C. E. Kelley of the University of Missouri has been appointed to an instructorship at Wentworth Military Academy, Lexington, Missouri.

Mr. C. G. Koch, who has been a student at Marquette University, has accepted a position with Illinois Bell Telephone Company, Chicago, Illinois.

Mr. S. B. Kramer, formerly a research assistant at the Polytechnic Institute of Brooklyn, has a position as an assistant project engineer with Sperry Gyroscope Corporation, Great Neck, New York.

Mr. H. C. Kranzer, previously a student at New York University, has been appointed to a research assistantship at the University.

Assistant Professor C. E. Langenhop of Iowa State College has been promoted to an associate professorship.

Dr. J. R. Lee, who has been an instructor at the University of Michigan, has been appointed to an associate professorship at the College of William and Mary.

Mr. Walter Littman, formerly a student at New York University, has been appointed to a research assistantship at the Institute for Mathematics of the University.

Professor E. R. Lorch of Columbia University has been appointed a Fulbright lecturer at the University of Rome.

Mr. H. J. Miller, who has been teaching at St. Patricks High School, Elizabeth, New Jersey, is teaching now at Linden High School, New Jersey.

Assistant Professor J. T. Moore of Georgia Institute of Technology has been appointed to an associate professorship at the University of Florida.

Professor Emeritus E. E. Moots of Cornell College has been appointed Lecturer at Occidental College for the academic year 1953–54.

Mr. S. I. Neuwirth, who has been affiliated with the Research Division of Schering Corporation, Bloomfield, New Jersey, has accepted the position of Biometrician with the Committee on Research of the American Medical Association, Chicago, Illinois.

Professor M. J. Norris of the College of St. Thomas is now a staff member of the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor M. M. Ohmer of Southwestern Louisiana Institute has been promoted to an associate professorship.

Mr. M. W. Oliphant of Georgetown University has been promoted to an assistant professorship.

Mr. T. D. Oxley, Jr., formerly a graduate student at Purdue University, has been appointed to a graduate assistantship at the University.

Mr. W. E. Pace of the University of Virginia has been appointed to an instructorship at Virginia Polytechnic Institute.

Miss Maria T. Pan, who has been a student at Chestnut Hill College, has been appointed Research Aide at New York University.

Mr. D. S. Park, previously with the National Bureau of Standards, Washington, D. C., has a position as a mathematician at the David Taylor Model Basin, Washington, D. C.

Mr. Eugene Park of Clemson College has been promoted to an assistant professorship.

Dr. M. O. Peach of the University of Notre Dame has been promoted to the position of Associate Professor of Engineering Mechanics.

Dr. W. H. Pell, who has been Head of the Department of Mathematics of the University of Kentucky, has been appointed to an associate professorship in the Graduate Division of Applied Mathematics, Brown University.

Mr. W. J. Pervin of the University of Pittsburgh has been appointed to a teaching assistantship at Carnegie Institute of Technology.

Dr. G. M. Petersen of the University of Arizona has been appointed to an instructorship at the University of Oklahoma.

Mr. T. J. Pignani of Loyola University has been promoted to an assistant professorship.

Assistant Professor C. J. Pipes, Jr., of Southern Methodist University has been promoted to an associate professorship.

Mr. A. O. Plait is engaged as an electrical engineer with the Admiral Corporation, Chicago, Illinois.

Mr. F. R. Pomilla of St. John's College has been promoted to the position of Assistant Professor of Physics.

Associate Professor Walter Prenowitz of Brooklyn College has been promoted to a professorship.

Mr. C. H. Reid, Jr., of Fort Lewis Agricultural and Mechanical College has been appointed Director of Admissions and Registrar of the College.

Mr. B. E. Rhoades, formerly a graduate student at Rutgers University, has been appointed to an instructorship at Lafayette College.

Mr. E. C. Rice of Monticello Agricultural and Mechanical College has been appointed to an assistant professorship at Central State College, Oklahoma.

Mr. A. St. C. Richardson, who has been a research engineer at Harvard University, has been appointed Research Fellow in the Department of Bacteriology and Immunology, Medical School of Harvard University.

Assistant Professor T. E. Rine of Illinois State Normal University has been promoted to an associate professorship.

Professor R. A. Rosenbaum of Reed College has been appointed to a professorship at Wesleyan University.

Assistant Professor H. L. Royden of Stanford University has been promoted to an associate professorship.

Professor L. P. Siceloff of Columbia University has been given the title of Professor Emeritus.

Mr. E. C. Smith, Jr., has been appointed to an instructorship at the University of Oregon.

Mr. J. L. Spenceley, previously of Alpena Junior College, Michigan, is teaching at Grand Haven High School, Michigan.

Mr. James Stone, who has been employed at the Oak Ridge National Laboratory, Tennessee, has joined the staff of Battelle Institute, Columbus, Ohio.

Mrs. Barbara F. Turner, formerly a student at Agnes Scott College, is teaching at Griffin High School, Georgia.

Professor J. A. Ward was on leave of absence from the University of Kentucky during the summer of 1953 and served as consultant at Holloman Air Force Base, New Mexico.

Associate Professor R. L. Westhafer of New Mexico College of Agriculture and Mechanic Arts has been promoted to a professorship.

Lt. G. C. Zader of the United States Naval Reserve has been appointed to an assistant professorship at The Citadel.

Dr. Henry Zatzkis of the University of Connecticut has been appointed to an assistant professorship at Newark College of Engineering.

Associate Professor E. A. Goodhue of the Missouri School of Mines and Metallurgy died on June 9, 1953.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-FOURTH SUMMER MEETING OF THE ASSOCIATION

The thirty-fourth summer meeting of the Mathematical Association of America was held at Queen's University and the Royal Military College, Kingston, Ontario, Canada, on Monday and Tuesday, August 31 and September 1, 1953, in conjunction with the summer meetings of the American Mathematical Society, the Canadian Mathematical Congress, the Institute of Mathematical Statistics, and the Econometric Society. A total of six hundred and thirty-seven

adults were registered, including the following two hundred and ninety-three members of the Association:

C. R. Adams, M. I. Aissen, Bess E. Allen, E. B. Allen, C. B. Allendoerfer, A. G. Anderson, R. D. Anderson, R. G. Archibald, Helen C. Arens, E. L. Arnoff, H. E. Arnold, Nachman Aronszajn, M. G. Arsove, H. T. R. Aude, Miriam C. Ayer, R. E. Barlow, I. A. Barnett, C. F. Barr, J. H. Barrett, R. C. F. Bartels, P. T. Bateman, W. R. Baum, L. D. Baumert, Samuel Beatty, P. R. Beesack, E. G. Begle, R. L. Beinert, Theodore Bennett, J. S. Bergen, R. H. Bing, D. W. Blackett, Shirley A. Blackett, David Blackwell, M. Isobel Blyth, H. F. Bohnenblust, Evelyn Boyd, C. B. Boyer, A. T. Brauer, G. U. Brauer, A. M. Bryson, F. J. H. Burkett, G. C. Bush, G. H. Butcher, James William Butler, E. A. Cameron, J. W. Campbell, K. H. Carlson, A. B. Carson, F. L. Celauro, Jeremiah Certaine, Abraham Charnes, Elsie T. Church, Randolph Church, R. V. Churchill, Paul Civin, Willie E. Clark, Helen E. Clarkson, D. E. Coffey, A. C. Cohen, Jr., L. W. Cohen, R. H. Cole, A. J. Coleman, J. B. Coleman, Esther Comegys, R. M. Conkling, T. F. Cope, A. H. Copeland, H. S. M. Coxeter, C. C. Craig, A. B. Cunningham, H. B. Curry, A. E. Danese, J. R. Davis, Jr., R. Y. Dean, Douglas Derry, W. A. Dolid, E. J. Downie, W. L. Duren, Jr., W. F. Eberlein, P. D. Edwards, M. P. Emerson, H. P. Evans, T. G. Evans, H. F. Fehr, H. H. Ferns, J. V. Finch, D. T. Finkbeiner, C. H. Fischer, G. E. Forsythe, J. S. Frame, Evelyn Frank, H. D. Friedman, Orrin Frink, Jr., Ruth M. Frisch, R. E. Fullerton, Ilse N. Gal, S. I. Gal, A. E. Gault, H. M. Gehman, Irving Gerst, B. C. Getchell, K. G. Getman, K. S. Ghent, Seymour Ginsburg, Sidney Glusman, Herbert Goertzel, Casper Goffman, Michael Goldberg, S. H. Gould, J. W. Green, Simon Green, F. L. Griffin, V. G. Grove, P. R. Halmos, H. W. Handsfield, W. C. Hansen, Bertha I. Hart, G. E. Hay, C. E. Heilman, E. R. Heineman, M. J. Hellman, Melvin Henriksen, J. G. Herriot, I. N. Herstein, Max Herzberger, Edwin Hewitt, T. H. Hildebrandt, R. V. Hogg, F. E. Hohn, D. L. Holl, T. R. Hollcroft, Aughtum S. Howard, W. A. Hurwitz, W. R. Hutcherson, Jack Indritz, R. L. Jeffery, L. W. Johnson, R. E. Johnson, F. E. Johnston, B. W. Jones, P. S. Jones, G. K. Kalisch, L. H. Kanter, Leo Katz, Hyman Kaufman, Dora E. Kearney, J. B. Kelly, J. R. F. Kent, D. E. Kibbey, E. C. Kiefer, F. T. Kocher, Jr., H. L. Krall, Max Kramer, Saul Kravetz, Solomon Kullback, O. E. Lancaster, N. D. Lane, C. E. Langenhop, E. H. Larguier, R. D. Larsson, H. L. Lee, F. C. Leone, Caroline A. Lester, W. J. LeVeque, F. A. Lewis, Florence Long, M. M. Lotkin, R. C. Lyndon, R. W. MacDowell, H. M. MacNeille, M. S. MacPhail, W. G. Madow, V. S. Mallory, Beckham Martin, Ceslovas Masaitis, K. O. May, N. H. McCoy, S. W. McCuskey, W. H. McEwen, S. W. McInnis, J. E. McLaughlin, E. J. McShane, Paul Meier, B. E. Meserve, C. E. Miller, E. B. Miller, Norman Miller, E. B. Mode, Harriet F. Montague, Mabel D. Montgomery, J. T. Moore, T. W. Moore, J. S. Morrel, D. C. Morrow, Thirza Mossman, C. W. Munshower, W. L. Murdock, Zeev Nehari, C. R. Newell, C. V. Newsom, Abba V. Newton, C. P. Nicholas, C. O. Oakley, E. G. Olds, E. M. Olson, J. C. Oxtoby, F. D. Parker, Sallie E. Pence, F. W. Perkins, G. M. Petersen, H. P. Pettit, C. G. Phipps, C. F. Pinzka, George Piranian, Everett Pitcher, J. C. Polley, I. R. Pounder, G. B. Price, Tibor Rado, J. F. Randolph, Ruth B. Rasmusen, G. E. Raynor, H. W. Reddick, R. M. Redheffer, P. K. Rees, R. F. Reeves, Eric Reissner, C. E. Rhodes, Audrey I. Richards, J. D. Riley, E. K. Ritter, H. E. Robbins, G. deB. Robinson, Louis Robinson, Arthur Rosenthal, Edward Rosenthall, M. F. Rosskopf, S. G. Roth, E. H. Rothe, J. P. Russell, Arthur Saastad, Charles Saltzer, Hans Samelson, R. D. Schafer, J. A. Schatz, Edith R. Schneckenburger, B. L. Schwartz, W. R. Scott, Edward Silverman, Annette Sinclair, Aubrey Henderson Smith, Robert Elijah Smith, D. O. Snow, W. S. Snyder, Vivian E. Spencer, Marion E. Stark, F. H. Steen, C. F. Stephens, B. M. Stewart, Irving Sussman, R. L. Swain, William Clare Taylor, G. H. M. Thomas, D. L. Thomsen, Jr., R. M. Thrall, H. S. Thurston, H. E. Tinnappel, Marian M. Torrey, A. W. Tucker, J. W. Tukey, H. L. Turrittin, J. L. Ullman, J. P. Van Alstyne, Helen E. Van Sant, J. E. Vollmer, T. L. Wade, Jr., R. W. Wagner, Eleanor B. Walters, Susie L. Ward, J. F. Wardwell, W. G. Warnock, J. V. Wehausen, C. P. Wells, J. G. Wendel, F. J. Weyl, A. L. Whiteman, G. T. Whyburn, W. L. G. Williams, Clement Winston, L. M. Winer, H. M. Zerbe, A. D. Ziebur, Antoni Zygmund.

Sessions of the Association were held on Monday afternoon, on Tuesday morning, and on Tuesday evening in Convocation Hall of Queen's University. President E. J. McShane presided at all sessions, except the joint session on Tuesday morning when the presiding officer was Professor W. L. G. Williams representing the Canadian Mathematical Congress. The second series of Earle Raymond Hedrick Lectures was delivered by Professor P. R. Halmos. The Program Committee for the meeting consisted of A. W. Tucker, Chairman, H. S. M. Coxeter, and Abel Gauthier.

FIRST SESSION OF THE ASSOCIATION

"Recent Developments of Mathematics in Canada," by Professor R. L. Jeffery, Queen's University.

"Some Mathematical Concepts," by Professor M. H. A. Newman, University of Manchester.

The Earle Raymond Hedrick Lectures: "Axiomatic Set Theory," Lecture I, by Professor P. R. Halmos, University of Chicago.

SECOND SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Axiomatic Set Theory," Lecture II, by Professor P. R. Halmos, University of Chicago.

Joint session with the Canadian Mathematical Congress:

"Integration in Archimedes," by Professor S. H. Gould, Purdue University.

"Analysis: Notes on the Evolution of a Subject and a Name," by Professor C. B. Boyer, Brooklyn College.

"Mathematics in Russian Universities between the Two World Wars," by Professor G. G. Lorentz, University of Toronto. (Presented by title.)

"The History of the Golden Section," by Professor H. S. M. Coxeter, University of Toronto.

THIRD SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Axiomatic Set Theory," Lecture III, by Professor P. R. Halmos, University of Chicago.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Reading Room of the Students' Memorial Union of Queen's University. Twenty-five members of the Board were present. Among the more important items of business transacted were the following:

The Board voted to approve the following dates for future meetings of the Association: December 31, 1953, at Johns Hopkins University, Baltimore, Maryland; August 30–31, 1954, at the University of Wyoming, Laramie, Wyoming; December 1954, at a place not yet determined; August 29–30, 1955, at the University of Michigan, Ann Arbor, Michigan; December 1955, at Rice Institute, Houston, Texas.

The Board voted to approve the recommendation of the Committee on the Earle Raymond Hedrick Lectures that Professor L. H. Loomis of Harvard University be invited to deliver the Hedrick Lectures at the 1954 Summer Meeting.

The Editor of the Monthly was authorized to publish ten 80-page numbers of the Monthly annually. In the past the December number has been a 64-page issue.

It was voted to cooperate with the National Council of Teachers of Mathematics in the publication of a high school mathematical journal by authorizing the President to appoint an Associate Editor of this journal and by appropriating five hundred dollars toward the expenses of the first year's publication.

On the recommendation of the Committee on Employment Opportunities, the Board voted to request the Editor of the Monthly to include a section in which paid advertisements of positions available would be printed. The Committee was also instructed to investigate the possibility of maintaining an employment register at meetings of the Association.

The report of the Committee on the Undergraduate Mathematical Program was received, the committee was discharged with a vote of thanks, and the President was authorized to appoint committees to carry out the recommendations of the report.

MEETINGS OF OTHER ORGANIZATIONS

Sessions of the American Mathematical Society began on Tuesday afternoon and continued through Saturday morning. The colloquium lectures: "On the existence and properties of certain singular integrals" were delivered by Professor Antoni Zygmund of the University of Chicago. Invited addresses were given by Professor Salomon Bochner of Princeton University and Professor Henri Cartan of the University of Paris.

The Summer Seminar and other sessions of the Canadian Mathematical Congress preceded the meetings of the Association. Lectures by Professors L. E. J. Brouwer and B. Segre were given on Monday morning. The Congress also sponsored a portion of the Tuesday morning session of the Association and the invited addresses by Professor Cartan.

The Institute of Mathematical Statistics and the Econometric Society held sessions from Tuesday through Friday. An invited address was delivered by Dr. H. O. Hartley of London.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of: Norman Miller, Chairman; L. W. Cohen, H. A. Elliott, H. M. Gehman, Israel Halperin, R. L. Jeffery, Percy Lowe, F. M. Wood.

Registration headquarters was in the Students' Memorial Union of Queen's University. Dormitory accommodations were available at the Royal Military College and in Ban Righ Hall of Queen's University from Sunday, August 30

until Saturday evening, September 5. Meals were served at Ban Righ Hall, the Royal Military College, and the Students' Union.

A welcoming reception was held on Monday evening in Wallace Hall of the Students' Union. A conducted tour of the city of Kingston and of Old Fort Henry was held on Tuesday afternoon. On Wednesday afternoon there was a choice between a motorboat trip from Gananoque among the Thousand Islands or a trip via a scenic route to the Sand Banks for swimming and a picnic supper. On Wednesday evening there was a showing of Canadian films in Grant Hall. A tea for the ladies was held on Thursday afternoon in Ban Righ Hall. On Thursday evening the members of the mathematical organizations were the guests of Queen's University and the Canadian Mathematical Congress at a theatre party. The International Players of Kingston presented the comedy "Goodbye Again" for the amusement of a large and enthusiastic audience.

At the theatre party, Professor J. W. Tukey presented a resolution of thanks on behalf of the visiting mathematicians, expressing gratitude for the felicitous arrangements and generous hospitality that marked this Kingston meeting as a successful and enjoyable one, long to be remembered by those present. Thanks were given to the administrations of Queen's University and of the Royal Military College, and especially to Professor Norman Miller as chairman of the Committee on Arrangements. The resolution also added an admiring tribute to the Canadian Mathematical Congress which has set such a brilliant record of achievements in its first eight years, and thanked the officers of the Congress for the important part they have had in holding this meeting in Canada.

HARRY M. GEHMAN, *Secretary-Treasurer*

REPORT TO THE ILLINOIS SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA OF ITS COMMITTEE ON THE STRENGTHENING OF MATHEMATICS TEACHING*

A. ANALYSIS OF THE PROBLEM AND STATEMENT OF PRINCIPLES

1. The nature of the problem and the task of the committee

The Problem as the Committee Sees It

On the basis of observation by its members and by others, the committee believes that there exists a powerful and dangerous trend against solid content in education, a trend which extends from the elementary schools through the high schools and which involves all subject-matter fields. It is important therefore to encourage teachers of mathematics, and of other subjects as well, to cooperate in a united effort to improve the quality of teaching and learning, and to improve the significance of the subject-matter taught in the schools.

* Presented to the Association at the Navy Pier, Chicago, Illinois, May 9, 1953.

The Brighter Side of the Picture

The committee recognizes, of course, that many teachers, administrators, school board members and other citizens are in fact doing splendid work on behalf of good education, frequently in the face of serious difficulties. To these it pledges all the support and cooperation it can give. Indeed, it hopes that its conclusions and recommendations will give voice to many of their problems and many of their aspirations.

The Committee's Task

Apart from a recognition of the seriousness and the scope of the problem, some aspects of which are detailed below, and apart from a recognition of the strength of the forces working against a sound educational program, the committee believes that its function is not primarily to join with the mounting roll of critics, but rather that its obligation is to develop—and, so far as possible, to implement—concrete, workable proposals. In short, the committee believes that its function is to help begin the task of reconstruction.

2. Problems relating primarily to the teacher

The Influence of Early Training of the Prospective Teacher

The importance of good teaching of mathematics, beginning at the elementary level and continuing through the high school, is emphasized by the fact that students learning poor mathematical methods in school will some day make poor teachers unless teacher training programs correct these unfortunate early learnings. It is a familiar experience of those connected with such programs that this is often a well-nigh impossible task. It is also a matter of experience that the teachers whose early training in mathematics was faulty frequently prefer teaching by rote to emphasizing principles and understanding. They will naturally place as little emphasis as possible on the subject they dislike. Many of them may not realize either the importance or the difficulty of teaching mathematics well. In fact, they may even believe that they have no mathematics teaching problems.

Influence of Educators and Administrators

A problem complementary to the one just described is posed by the type of professional educator or public school administrator who declares that penmanship, or the alphabet, or grammar, or mathematics is not important in modern education. The influence of such educators is demoralizing to the already weak teacher and is discouraging to the good teacher who has to fight for his subject against those whose authority is greater. Their influence can only serve to promote inferior teaching or outright neglect of the more vital but also more difficult subject matters. Again, it is a matter of observation that the "core" and "life adjustment" programs have played into the hands of such individuals,

with the result that basic disciplines, such as mathematical training for example, are neglected or, in some cases, are omitted entirely. The consequence of such neglect is disastrous for many an able student and unless corrected, it will inevitably be disastrous for the nation. The influence of the educator and the administrator ought to be such as to eliminate vague and ill-defined aims and to encourage the thorough teaching of fundamentals. In this way and only in this way can the educators, the teachers and the schools best serve the interests of the students and of the nation.

The Professional Training of Teachers

In regard to the preparation of teachers, the committee believes that, on the undergraduate level, the segregation of prospective teachers into colleges of education, with a consequent emphasis on courses in "education," is a mistake. The prospective teacher, as much as other students if not more, needs a well-rounded, liberal education. Moreover, experienced elementary and secondary teachers should, we believe, play the maximum possible part in the professional training of the prospective teacher. We believe that specialization in "education" as a major subject should be reserved to the graduate levels.

Abdication of the Scholars

It must be admitted, however, that the subject matter fields have lost the prospective teachers to the colleges of education largely by default. In past decades we pretended, in our scholarly isolation, that a genuine concern with the problems and the preparation of the elementary and secondary teacher was beneath us. But it is one of the laws of society that when those who by rights should assume leadership refuse to accept the responsibility, others step into assume their tasks. In the current and mounting cry over the state of American education, one thing is conspicuous: there is too little self-examination. That many who do not comprehend the true nature and purpose of education are in the position of directing it is the result of the scholars' abdication of one of their most precious responsibilities. Now, when it is almost too late, we perceive the magnitude of the catastrophe.

Graduate Mathematics Courses for Teachers

The loss by default penetrates even into the graduate level, where there is a definite need for special, non-Ph.D. courses providing material that is interesting, broadening, helpful and especially selected for teachers. It is important that such courses be offered in summer sessions where they could be far more profitable to the returning teacher than are the too commonly repetitious courses offered by colleges of education, courses which are taken mainly because they can be passed satisfactorily by someone out of contact with the Ph.D.-type of graduate work. The experiment for college teachers beginning at Boulder, Colorado, this summer may well provide suggestions for similar programs for high

school and elementary teachers, programs which would enable them to meet certification requirements while learning something both inspirational and valuable.

Workshop for Teachers

The committee also believes that there is a need for more workshops for teachers at which teachers themselves would be present and discuss the problems and difficulties of teaching elementary and high school mathematics. Adequate provision should always be made for discussion, and the time allotted for this should not be usurped by the discussion leader, as is all too frequently the case. In large measure, these workshops should be small, localized meetings of short duration, located within easy access of the teachers for whom they are intended. Such meetings would provide an opportunity for beginning teachers to contact more experienced teachers for advice and encouragement. At the present time, many a promising young teacher abandons the profession because of discouragement during the initial years. The orientation and aid obtainable from other teachers at workshops such as are here proposed could serve to prevent many of these losses of teachers to other fields of activity. Indeed, this advisory function of experienced teachers ought to be organized into a systematic scheme for getting aid to the beginning teachers who need it.

An important part of such workshops could be concurrent meetings for high school students, designed to give them orientation and inspiration in mathematics.

These local workshops would of course be intended to supplement rather than to supplant the major meetings drawing on wider areas, such as are being held from time to time by various colleges and organizations.

Fellowships, Scholarships and Stipends

As a variation of the workshop plan, the committee believes that some foundation, or even state departments of education, would do well to finance a system of traveling fellowships for outstanding teachers, who would thus be enabled to visit many schools to offer advice and encouragement, and through broad observation and experience, to make concrete proposals for the improvement of the teaching situation as a whole.

Another method of improving the performance of teachers is through scholarships and stipends which aid underpaid members of the profession to avail themselves of the benefits of various special workshops and training programs in existence throughout the country. The *Mathematics Teacher* is to be commended for its policy of circulating information about such programs and stipends. Certainly the number of such stipends ought to be increased as a means of recognizing promise and of improving the standards of the profession. We believe that school boards should recognize this and, whenever possible, finance such opportunities for their teachers.

Salaries and Promotions

Finally, the committee recognizes the need for higher salaries in the teaching profession, salaries adequate to draw competent people to the profession and to keep them there. But it believes also that these salaries should be based on training in subject-matter fields and on demonstrated teaching competence, rather than on the completion of more courses in "education," as is frequently the case at present.

3. Problems relating primarily to the student

The Problem of Proper Guidance

Assuming competent instruction, the primary problem in the case of the student is one of proper guidance. We believe that much advice to the effect that mathematics is not necessary or desirable is based on the misconception that the subject is inherently difficult and dull for most students. However, the phenomenal success of recent mathematical TV programs, and the widespread and perennial interest in mathematical puzzles and riddles is proof of a greater latent interest in mathematics than is ordinarily suspected of the general run of mankind. This interest can be kept alive and developed in students only by good teaching and by proper emphasis on the importance of mathematics in human affairs. It is therefore a duty of the mathematics teacher and the administrator properly to inform students, patrons, and those responsible for guidance, of the vital significance of the subject.

The Importance of Mathematics in Modern Life

The fact of the matter is that in every area of scientific investigation today, as well as in many types of skilled labor, a certain body of mathematical skills is required for effective participation. Moreover, these requirements are rapidly increasing and will continue to do so as our civilization becomes increasingly technical. Every student and parent should know this, and moreover, every high school student should continue his mathematical training to the full extent of his ability. Whether or not he goes on to college—and this is something he may not be sure of himself—this training will serve him in good stead if it is sound material and is well taught. What the colleges, science, industry and above all the vocation of citizenship need and cry for today is the help of thinkers, not dilettantes or robots.

Aids to Proper Guidance

There are various aids available for a proper guidance program. There is a guidance bulletin prepared by the National Council of Teachers of Mathematics, there is the widely circulated bulletin on the mathematical needs of prospective engineering students prepared by the University of Illinois, there are circulars prepared by industry, such as General Electric's *Why Study Mathematics?*, and there are helpful articles in such journals as the *Mathematics Teacher* and the

American Mathematical Monthly. A pamphlet listing these aids and suggesting proper methods in mathematics guidance would be of great value if it could be circulated extensively at little cost among educators, patrons, students and those responsible for guidance.

Guidance Through Informal Lectures

The committee is aware that not all guidance is planned as such, but may rather be a byproduct of other experiences. As an example of this, the committee has had the opportunity to observe how effectively good popular lectures on mathematics can represent its importance in the educational process as well as in modern life and thought. We therefore urge mathematics teachers and interested laymen to utilize every opportunity to speak in behalf of the subject, whether it be to groups of students or to groups of patrons.

The Responsibility of the Teacher

No amount of proper guidance can solve the problem, of course, in the absence of effective and inspirational teaching. Many teachers have found that striking exhibits, properly used, will create interest in and convey the importance of mathematics. There is unlimited room for ingenuity here. Frequently topics commonly presented only in a formal way can be made both meaningful and inspirational by a few simple indications of how one would use these ideas in the factory, the laboratory, the market place, or in the understanding of nature. Ideas for enrichment materials of this kind are to be found in the professional periodicals, with which every teacher should be thoroughly familiar. Every teacher ought also to feel an obligation to share with other teachers good ideas which may not be widely known, through the medium of such periodicals. These observations point up to the extent to which the problem reverts to the classroom teacher. Sound material, inspirationally taught, is one of the best means of advertising our subject and represents the greatest contribution to good education we can possibly make.

4. Problems relating to schools and curricula

Correspondence Courses for High School Students

The committee feels a great concern for the high school student who needs and desires adequate mathematical training but whose school cannot or will not offer the necessary courses. The committee believes that genuine intellectual ability is so precious a quality that it ought never be allowed to suffer for lack of the opportunities necessary for proper development. We believe therefore that the universities should follow the lead of such schools as the University of Wisconsin in developing and advertising energetically a program of correspondence courses in those secondary school subjects which are frequently neglected.

One effect of advertising such a program would of course be such as to make students more fully aware of the importance of mathematics in their prepara-

tion. It would also make school boards more fully aware of their responsibility for providing proper curricular opportunities. In the long run, one would therefore expect such a program to be self-liquidating, which is as it should be.

The Problem of a Good Curriculum

It is a matter of observation that guidance, instruction and administration in certain high schools is of such a high calibre that the graduates of these schools do outstandingly well in employment requiring technical skills as well as in college work and beyond. These are schools in which thorough teaching of fundamentals has been emphasized. This fact alone is evidence that the critics of modern education are dealing with genuine issues. Moreover, it emphasizes the need for making teachers, school officials, and patrons aware of the statistical flaws in the now notorious *Eight-Year Study*, flaws which render its conclusions of little value. The fact of the matter is that as yet, nobody has demonstrated that a properly balanced program of subject-matter courses, with the emphasis on methods of creative and critical thinking, is not the best preparation for college work or for life.

The Problem of Maintaining High Standards

In view of the above observations the committee wishes to go on record as favoring the subject-matter type of curriculum as opposed to the core type of program, specific subject-matter requirements for high-school graduation and for college entrance, and mathematics for all those able to comprehend it. This does not mean that the committee is opposed to changes in content of courses, in methods of instruction, or in methods of evaluating achievement. It does mean that the committee is opposed to such changes as imply lowered standards of scholarship and accomplishment. Certainly educational research ought to enable us to improve these standards rather than to urge us to lower them. When it does the latter, it runs counter to the realities of our times, and is based on false premises, faulty observation or errors of logic. It is indeed not the peculiar whim of the professors, nor of the college administrators, that standards be kept high; it is the demand of life itself. The real life-adjustment is no retreat into a core of mediocrity, but rather an honest facing of the fact that knowledge and the wisdom to use it constructively are the essence of survival.

5. The problem of cooperation

There are certain inherent dangers involved in an attempt to implement convictions such as have been expressed here. For example, the scholars have so long ignored their responsibilities to the elementary and secondary teachers that a rift has developed between the elementary and secondary schools on the one hand and the colleges on the other. Great care must therefore be exercised to make certain that what are intended to be constructive efforts may not be interpreted as unwarranted criticism or interference.

Another danger is that the perils and needs of the times may not merely permit the overdue strengthening of mathematics and science in the elementary and secondary schools to take place, but may in fact result in an overemphasis on these subjects, to the neglect of the humanities. We have as much or more reason to protest the creation of scientific robots as to protest the neglect of scientific subjects in the curriculum.

As a matter of fact, the problem is more than just how much arithmetic, or how much history, the pupil should have. The issue is rather this: Shall we or shall we not give our children both the fundamental knowledge and the training in creative and critical thinking that they need for intelligent and effective participation in a democratic society? This is a problem of the utmost seriousness! The time has come when we must join forces at all levels and in all subject matter fields, for the battle to guarantee our children the educational opportunities they need and deserve is destined to be a long, hard one, and it is only by means of the highest degree of cooperation that it can be won.

B. ACTIVITIES OF THE COMMITTEE

In addition to drawing up the preceding analysis and statement of principles, the committee has listened to reports from the following members of the University of Illinois staff:

(a) Professor B. E. Meserve, who outlined the plans and progress of a committee at the University of Illinois which is developing experimentally at the University High School a four-year, unified high school mathematics program which will be terminal after two years for students not preparing for a technical career, but which will continue through all four years for students whose interests or plans demand a more adequate mathematical training;

(b) Professor S. S. Cairns, who described the interests of the Research Education Division of the National Science Foundation which resulted in its setting up this summer's program at Boulder, Colorado, and which may result in its cooperating with efforts to provide similar opportunities for high school teachers; Professor Cairns also discussed the concern of the Scientific Manpower Commission for good college programs, indicated its possible interest in secondary programs and reported on the planning session for the Allerton House Conference on Education;

(c) Professor R. E. Pingry, who outlined plans of the joint MAA-NCTM committee for the study of teacher training programs in mathematics; and

(d) Dean Stanley H. Pierce, who outlined plans of the Secondary Schools Committee of the American Society for Engineering Education for the study of means of closer articulation of high school and college programs in mathematics.

Besides informing itself in this way on the activities of other committees having similar concerns, with the object of instituting cooperation and avoiding duplication of effort wherever possible, the committee has conveyed its concerns to other groups and individuals, including the following:

(i) to the Illinois Council of Teachers of Mathematics, information concerning our purposes and a request for an expression of support

(ii) to the National Council of Teachers of Mathematics, lists of proposed subjects for yearbooks and pamphlets, and support of a proposal for a national high school mathematics publication

(iii) to Dr. Everett Welker a proposal that he prepare for publication his statistical criticism of the *Eight-Year Study*, given before this group last year

(iv) to the editors of the *Mathematics Teacher* and the *American Mathematical Monthly* a proposal that they consider publishing critical articles on the *Eight-Year Study*

(v) to the Provost of the University of Illinois, an offer to provide all possible assistance to the mathematical sub-committee helping to prepare for the Allerton House Conference on Education

(vi) to the other committees with concerns similar to ours, offers of all possible cooperation

(vii) to Senator Charles Clabaugh of the School Problems Commission, an expression of our conviction that school boards should be required by law to provide, at their expense and by correspondence if necessary, advanced high school mathematics courses for all students whose future plans require them, and to allot study time for such courses from the student's school day.

Replies to these communications so far received have been uniformly favorable.

C. RECOMMENDATIONS OF THE COMMITTEE

In order to implement further its conclusions, the committee recommends to the Illinois Section of the Mathematical Association of America the following actions:

(a) that the Illinois Section recommend to the Mathematical Association of America that it appoint a Committee on Cooperation with Industry to capitalize on industry's known willingness to invest money in advertising the need for good mathematical and scientific training at the high school level and in workshops for high school teachers, designed to inform them of ways in which mathematics is applied to practical problems

(b) that the Illinois Section and the Illinois Council of Teachers of Mathematics explore the possibility of cooperating in the organization in Illinois of an annual summer workshop for high school teachers

(c) that the Illinois Section recommend to the MAA that it offer the NCTM aid, support and cooperation in the proposed launching of a national high school mathematics periodical

(d) that the members of the Illinois Section interested in teacher training be urged to offer to prepare for the NCTM, pamphlets on such topics as "How to Organize a Workshop," "How to Make a Good Assignment,"

and others of the many worthy topics for which the NCTM has been unable to find authors

(e) that the Illinois Section urge its members to join the Illinois Council and the National Council of Teachers of Mathematics as an expression of support of efforts of these organizations to improve the teaching of secondary mathematics

(f) that the Illinois Section recommend to the Mathematics Department of the University of Illinois that it prepare and arrange for the distribution of a companion to the bulletin on the mathematical needs of prospective engineers, the new bulletin being designed to list the mathematical needs of students planning to enter fields of study other than engineering

(g) that the Committee for the Strengthening of Mathematics Teaching be made, officially as well as in effect, a joint endeavor of the Illinois Section and the Illinois Council

(h) that the Illinois Section consider a moderate increase in its dues in order to be able to defray reasonable expenses of such committees as it may appoint, and finally

(i) that the Illinois Section approve this report and authorize its circulation among concerned members of
the several mathematical organizations
the public and the press
the various agencies of the University of Illinois
the legislature and committees thereof and
the United States Office of Education.

Respectfully submitted,

MARY ENTSMINGER
Laboratory School
Southern Illinois University
MARTHA HILDEBRANDT
Proviso Township High School
FRANZ E. HOHN (Chairman)
University of Illinois
ANICE SEYBOLD
North Central College
HENRY SWAIN
New Trier Township High School

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-second annual meeting of the Illinois Section of the Mathematical Association of America was held at the University of Illinois, Navy Pier, Chicago, Illinois, on Friday afternoon and Saturday forenoon, May 8 and 9, 1953. Professor H. G. Ayre, Chairman of the Section, presided at all sessions.

There were seventy-nine in attendance, including the following fifty-eight members of the Association:

Beulah M. Armstrong, H. G. Ayre, Imogene C. Beckemeyer, Winifred V. Berglund, A. H. Black, R. L. Blair, Angeline J. Brandt, B. K. Brown, J. R. Brown, S. S. Cairns, Laura E. Christman, John Christopher, H. S. Clair, Flora Dinkines, F. W. Donaldson, I. K. Feinstein, Evelyn Frank, A. E. Gault, G. D. Gore, L. M. Graves, A. E. Hallerberg, M. C. Hartley, E. W. Hellmich, E. H. C. Hildebrandt, Martha Hildebrandt, F. E. Hohn, Rose L. Hornacek, C. A. Jacokes, E. C. Kiefer, Rose Lariviere, A. O. Lindstrum, Jr., W. C. McDaniel, A. W. McGaughey, B. E. Meserve, R. J. Mihalek, E. B. Miller, M. G. Moore, Elsie C. Muller, H. E. Nelson, Grace M. Nolan, F. S. Nowlan, Margaret Olmsted, C. E. Olsen, T. B. Ondrak, Gordon Pall, J. W. Peters, R. E. Pingry, Ruth B. Rasmusen, Haim Reingold, T. E. Rine, L. A. Ringenberg, M. Anice Seybold, A. T. Street, B. R. Ullsvick, L. R. Van Deventer, Arnold Wendt, Charles Craig Wilson, Alice K. Wright.

At the business meeting the following officers were elected for the coming year: Chairman, Professor M. C. Hartley, University of Illinois, Navy Pier, Chicago; Vice-Chairman, Professor Rothwell Stephens, Knox College; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University. The Secretary-Treasurer's report was read and accepted. Then the retiring Sectional Governor reported on the work of the Board of Governors for this year.

The following program was presented:

1. *Content and function of pre-calculus mathematics for the preparation of secondary teachers*, by Professor B. R. Ullsvik, Illinois State Normal University.

The revision of content only will not improve pre-calculus mathematics for purposes of teacher education because of the importance of the teacher in inspiring students both for learning and for becoming good teachers. Accepting that "Any program of training teachers must be tied into what they teach," the undergraduate courses in mathematics should prepare for the teaching of high-school "general mathematics," as well as the traditional courses in algebra and geometry.

The present trends of student enrollments indicate that an increase in numbers can be expected and no appreciable change in the quality of students can be hoped for. The typical pre-calculus courses do not adequately prepare for the teaching of Euclidean geometry, and a review of high-school geometry would be more appropriate than merely more theorems in college geometry. There is much need for further investigation concerning the value of integration of pre-calculus courses rather than teaching compartmentalized college algebra, analytics and college geometry.

2. *Contribution of related areas to the preparation of teachers of secondary mathematics*, by Professor A. W. McGaughey, Bradley University.

The paper called attention to some studies which had been made in this area, pointing out that secondary school principals, teachers of mathematics on the secondary level and university professors all recommended that the candidate for a teaching certificate in secondary mathematics should obtain as broad a cultural background as possible, including economics, sociology and ethics as well as the physical sciences. The university professors in their report in the May, 1935 issue of this MONTHLY recommended that these courses be taken in preference to advanced work in mathematics or in theory of education beyond the legal requirements.

The author believes that the secondary teacher needs many illustrative examples during his beginning years and should be helped in gathering them. He suggested the possibility that this Association might sponsor such an endeavor.

3. *What the teacher training institutions can do to improve the teaching of secondary mathematics*, by Professor R. E. Pingry, University of Illinois.

Despite the high probability for a critical shortage of qualified mathematics teachers in secondary schools in the next ten years, colleges should continue to hold high standards for teacher education programs. Colleges should develop better selection procedures so poor prospects for teaching will not be certified. The mathematics education of teachers should be evaluated, and improved student teaching programs developed. More attention should be given to the problem of teaching junior high school mathematics and to developing in-service training programs for mathematics teachers who are already teaching.

4. *An adequate master's degree program for teachers of secondary mathematics*, by Professor E. H. C. Hildebrandt, Northwestern University.

The program is based on the questions most frequently raised by teachers with secondary school teaching experience. Requests include advanced work in mathematics related to the content of the secondary courses including a better understanding of the foundations of the calculus, an introduction to the fields of higher mathematics, and a systematic approach to problem-solving as required by the mathematics student, engineer, scientist and industrial mathematician. Problems of the mathematics curriculum and better teaching techniques in mathematics also require further study. Finally, are these courses alone adequate? Can the written thesis actually provide for additional training needed in this profession?

5. Address at dinner meeting: *Abstractions*, by Professor S. S. Cairns, University of Illinois.

The speaker planned an entertaining, non-technical talk that would interest all attending the dinner by comparing some general abstractions with some encountered in mathematics.

6. *What the high school principals and superintendents are doing to strengthen mathematics programs in the schools*, by Dr. N. E. Watson, Northfield Township High School, Northbrook, introduced by the Secretary.

Dr. Watson stressed the fact that mathematics is a language and that it must so be taught. He questioned the preparation of mathematics teachers and their ability to present the subject so that pupils could see its importance and functional aspects. He stressed the development of new and better instructional materials, different attitudes toward the place of mathematics among the other courses offered and a sense of humor in its presentation. He urged Master's degrees in mathematics for teachers of the subject.

He urged more mathematics, rather than less, for high school pupils but at their level of need and understanding. He said high schools are doing much counseling after testing programs in mathematics. He urged better courses, materials and points of view. He urged all mathematics teachers to consider themselves teachers of the language of mathematics.

7. *Report of the committee on the strengthening of mathematics in secondary schools*, by Professor F. E. Hohn and Committee.

This report is published in full in this issue, page 652.

E. C. KIEFER, *Secretary.*

THE MAY MEETING OF THE NEBRASKA SECTION

The twenty-ninth annual meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska, Lincoln, Nebraska, on May 2, 1953. Mr. H. W. Becker presided at the session.

Thirty-nine persons attended the session including the following twenty-five members of the Association:

M. A. Basoco, H. W. Becker, W. C. Brenke, C. C. Camp, N. A. Court, H. M. Cox, Morris Dansky, H. W. Doss, Jr., J. M. Earl, M. G. Gaba, C. B. Gass, Edwin Halfar, A. L. Johnson, Jr., W. G. Leavitt, W. T. Lenser, E. J. Lowry, R. L. Moenter, Theodora S. Nelson, T. A. Newton, C. R. Perisho, Florence E. Pool, H. B. Ribeiro, Harry L. Rice, Lulu L. Runge, A. J. Tingley.

Officers elected at the meeting were: Chairman, Professor C. C. Camp, University of Nebraska; Vice-Chairman, Mr. H. W. Becker, Station WOW-TV, Omaha; Secretary-Treasurer, Professor Edwin Halfar, University of Nebraska.

Professor N. A. Court of the University of Oklahoma was the guest speaker.

The following papers were presented:

1. *A note on convergent monotone series*, by Dr. T. A. Newton, University of Nebraska.

Several excellent papers have been written within the last ten years showing how a theorem of de La Vallee Poussin concerning a generalization of the Cauchy condensation test can well serve as a basis for a study of infinite series of positive terms. In particular, this theorem was used to develop more general and more powerful convergence and divergence criteria. In this paper, the theorem served as the basis for the proof of a theorem concerning the behavior of the terms of a convergent monotone series.

2. *A report on the mathematical training of entering University of Nebraska freshmen*, by Professor H. M. Cox, University of Nebraska.

3. *Applications of matrix theory to differential equations*, by Professor W. G. Leavitt, University of Nebraska.

In this paper some examples were given illustrating the use of matrix notation in setting up systems of linear differential equations. The advantages of this notation lie not only in its conciseness, but also in the possibility it gives for drawing on known theorems from the theory of matrices.

4. *Desargues' strange theorem*, by Professor N. A. Court, University of Oklahoma.

5. *Unsolved problems of arithmogeometry*, by Mr. H. W. Becker, Station WOW-TV, Omaha.

Do there exist: (a) Parallelotopes of n integer sides such that the square roots of the sums of the squares of any m are also integers, $n=3$ or greater (see Dickson's *History* II, xiv)? (b) The previous, with $m=2$, and $n=4$ or greater (hypercuboids, see M. Kraitchik, *Scripta Mathematica*, 1945, p. 317)? (c) Orthoschemes (regular graphs, see H. S. M. Coxeter, *Regular Polytopes*, p. 137) of n points, $(n, 2)$ lines, and $(n, 3)$ Pythagorean triangles, $n=5$ or greater? Parametric solutions were discussed for sets of integers almost meeting the desired conditions.

6. *On the formulation of the definition of determinant for linear mappings*, by Professor H. B. Ribeiro, University of Nebraska.

7. *Remarks on teaching of geometry*, by Professor N. A. Court, University of Oklahoma.

EDWIN HALFAR, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The May meeting of the Wisconsin Section of the Mathematical Association of America was held at Mount Mary College, Milwaukee, Wisconsin, on May 2, 1953. Mr. A. C. Moeller presided in the absence of the chairman, Professor R. H. Bing.

Ninety-two persons attended the meeting, including the following thirty-eight members of the Association:

R. H. Bardell, E. H. Batho, Leon Battig, L. J. Berner, W. W. Bigelow, C. C. Braunschweiger, Leonard Bristow, R. C. Buck, E. G. H. Comfort, Rev. L. A. V. DeCleene, W. E. Deskins, H. P. Evans, F. R. Harding, R. C. Huffer, Rev. M. L. Jautz, J. F. Kenney, K. L. Leverance, A. P. Loomer, C. C. MacDuffee, Morris Marden, A. E. May, Genevieve L. Meyer, A. C. Moeller, Marianne S. Otto, O. E. Overn, G. A. Parkinson, H. P. Pettit, J. G. Renno, Jr., R. E. Schwartz, Sister M. Elizabeth, Sister Mary Felice, Sister Mary Petronia, Abraham Spitzbart, J. V. Talacko, W. J. Thomsen, R. D. Wagner, L. F. Wahlstrom, Louise A. Wolf.

The following officers were elected for the year 1953–54: Chairman, Dr. L. F. Wahlstrom, State Teachers College, Eau Claire; Secretary-Treasurer, Sister Mary Felice, Mount Mary College; Chairman of the Program Committee, Professor Abraham Spitzbart.

The following papers were presented:

1. *Geometry and modern algebra*, by Dr. C. W. Curtis, University of Wisconsin, introduced by Professor R. H. Bing.

The familiar concepts of algebraic curves and surfaces lead in a natural way to the definition of an algebraic manifold in n-dimensional space over a given field K. It was shown that geometrical questions concerning algebraic manifolds could be replaced by algebraic questions concerning ideals in the ring $K(x_1, \cdots, x_n)$ of polynomials in n variables with coefficients in K; in particular the purely geometric concept of an irreducible algebraic manifold can be replaced by the concept of a prime ideal in $K(x_1, \cdots, x_n)$. It was indicated how this transition can be accomplished by making use of two fundamental results first proved by Hilbert: the basis theorem for polynomial ideals, and the zero theorem.

2. *Variations on a theme*, by Professor H. P. Pettit, Marquette University.

Starting with a vector space, we assume the norm as not necessarily a positive definite quadratic form. Then the Schwarz inequality may not hold. The minimal cone, found by equating the norm to zero, may be real. A surface with constant radius turns out to be a quadric asymptotic to the minimal cone. Two vectors are called normal to each other if they are conjugate relative to the minimal cone. Using vector methods, one easily shows that the properties of perpendiculars in ordinary geometry have analogous theorems in this system, some of which were new to the writer. This variation from standard procedure gives the vector methods added power in the study of geometry.

3. *The concept of area in plane geometry*, by Professor C. C. MacDuffee, University of Wisconsin.

Hilbert defined the phrase "of equal area" to mean that two plane polygons could be decomposed into a finite number of triangles which were respectively congruent to one another in pairs. The relationship "of equal area" is determinative, symmetric, reflexive and transitive. Area, then, is a concept which we introduce in order to transform the equivalence relationship "of equal area" into an actual equality.

4. *Operational research*, by Professor J. V. Talacko, Marquette University.

Operational Research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control. It is a new science with mathematics, especially probability and statistics, as a main tool. It may be divided into two parts: military and civil. Both branches are in the stage of development and expansion. It is expected that the growth of this new mathematized science will be similar to that of statistics. The Society of Operational Research was organized in 1952 and has over 400 members. The mathematicians with ability for research, with background in the sciences of physics, chemistry, biology, psycometrics, now have open a new field of activity, employment and growth.

5. *A mathematically designed painting*, by Miss Audrey Reiff, Mount Mary College, introduced by Sister Mary Felice.

Periodicity is the habitual recurring at regular intervals of a definite unit. The synchronization of a units taken b at a time with b units taken a at a time produces a symmetrical pattern known as a derivative period. By properly selecting such a period and translating it as a series of lines whose directions are governed by proportionate angles, a good design is obtained. Through superimposition, the addition of 180° arcs, and a few realistic lines, a picture can be obtained which is mathematical in its basis and artistically correct in design.

6. *A topic in the theory of stochastic processes*, by Dr. E. A. Immel, University of Wisconsin, introduced by Miss Louise A. Wolf.

7. Panel discussion: *Opportunities for Mathematicians in Industry*; Moderator: Mr. W. W. Bigelow, A. O. Smith Corporation; Members of panel: Mr. Koth, A. O. Smith Corporation, Mr. W. M. Snell, Northwestern Mutual Life Insurance Company, Mr. Robert Richardson, President of Wisconsin Society of Certified Public Accountants.

The mathematician in industry is usually employed either in computing, in statistical quality control, or as a consultant on engineering problems. The first category provides opportunities for recent graduates in positions as calculators or engineering aides, with present starting salaries comparable to those of engineers. Consulting work is probably the most desirable, and certainly provides the most rewarding positions. Mr. Koth, an experimental engineer at A. O. Smith Corporation, emphasized the importance of an academic background in engineering subjects in any of these positions. In general, salaries in industry are greater than in government work and substantially higher than in education.

Mr. Snell pointed out that an actuary cannot operate without a thorough knowledge of mathematics. However, he is essentially a business man who plays an important part in developing general company policies. One qualifies as an actuary by passing the examinations of a recognized actuarial body. The field is not crowded. With more than 700 life insurance companies, plus filling the needs of government and consulting firms, the opportunities are limitless. There are about 780 Fellows of the Society of Actuaries, but many more are needed.

Mr. Richardson discussed the educational and experience requirements for the Certified Public Accountant's certificate. These include mathematics at the high school level and a commerce course at a university with an accounting major. The importance of arithmetic, algebra, geometry and trigometry in the training program should be stressed, while many other mathematical subjects provide important background and increase the analytical power of the candidate. An accountant must be accurate and quick and literally live with figures. A sure road to success in this profession is the use of mental facilities rather than frequent recourse to automatic calculators.

SISTER MARY FELICE, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The seventh annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at Montana State University, Missoula, Montana, on June 19, 1953, in conjunction with the four hundred ninety-third meeting of the American Mathematical Society. Professor R. A. Rosenbaum, Chairman of the Section, invited Professors Harold Chatland, R. D. James, and S. G. Hacker to preside at the afternoon session.

Fifty-seven persons were in attendance, including the following forty-three members of the Association:

P. M. Anselone, T. M. Apostol, M. G. Arsove, D. O. Banks, R. A. Beaumont, J. L. Botsford, J. L. Brenner, L. G. Butler, Harold Chatland, P. A. Clement, C. M. Cramlet, R. Y. Dean, D. B. Dekker, Paul Erdös, R. M. Gordon, S. G. Hacker, Mary E. Haller, C. A. Hayes, Jr., H. H. Irwin, R. D. James, J. M. Kingston, M. S. Knebelman, R. B. Leipnik, A. T. Lonseth, R. E. Lowney, J. E. Maxfield, A. S. Merrill, W. E. Milne, Leo Moser, W. M. Myers, Jr., Ivan Niven, Andrewa R. Noble, Ina S. Olson, T. G. Ostrom, C. A. Pursel, R. A. Rosenbaum, Louise J. Rosenbaum, A. J. Smith, W. M. Stone, J. R. Vatnsdal, D. J. Walker, R. M. Winger, F. H. Young.

A business meeting was held in the evening at which the following officers were elected: Chairman, Professor Harold Chatland, Montana State University; Vice-Chairman, Professor Ivan Niven, University of Oregon; Secretary-Treasurer, Professor J. M. Kingston, University of Washington. A Program Committee for the 1954 meeting was appointed, consisting of Professor L. B. Williams, Chairman, and Professors M. G. Arsove, S. G. Hacker, T. E. Hull, and E. S. Keeping.

The afternoon session consisted of the following invited hour address, three twenty-minute papers, two ten-minute papers, and a symposium by a panel of three speakers:

1. *The distribution of quadratic residues*, by Professor Leo Moser, University of Alberta. (By invitation.)

This paper was a historical outline of the theory of distribution of quadratic residues. After presenting the contributions of Gauss and Dirichlet, three main types of theorems were discussed. The first concerned existence and frequency of blocks of residues and of non-residues. Here the results of Jacobsthal, Bennet, Davenport, A. Brauer, and Perron were presented. Secondly, results were given showing that the least non-residue is small, but sometimes not very small. These results are due to Nagell, Vinogradoff, Brauer, Pillai, Ankeny and others. Finally, it was shown that in certain small ranges of integers, at least a certain fixed fraction of the numbers are residues and at least an equal fraction are non-residues. These results were obtained by Vandiver, Redei, and the author.

2. *A risk function approach to the problem of tolerance limits*, by Professor Bernard Ostle, Montana State College, introduced by the Secretary.

Given a function $f(x) \geqq 0$ for $-\infty \leqq x \leqq \infty$ such that $\int_{-\infty}^{\infty} f(x)dx = 1$, defining $F(x) = \int_{-\infty}^{x} f(x)dx$, and having a random sample of n values from the population specified by $f(x)$, the problem is to find two functions of the sample values, say $L_1(x_1, x_2, \cdots, x_n)$ and $L_2(x_1, \cdots, x_n)$ such that $P\{F(L_2) - F(L_1) \geqq \beta\} = \alpha$ where $0 \leqq \alpha \leqq 1$ and $0 \leqq \beta \leqq 1$. The question of most interest is to determine the minimal value of n which will satisfy the equation $P\{F(L_2) - F(L_1) \geqq \beta\} = \alpha$ once the forms of L_1 and L_2 have been decided upon. One approach is to consider a weight function $W(L_1, L_2)$ and a cost function $C(n)$, and to determine n so as to minimize the risk involved, that is to minimize

$$r(n) = \int \cdots \int W(L_1, L_2) \prod_{i=1}^{n} f(x_i)dx_i + C(n).$$

This has been done for certain choices of L_1, L_2 and W, and the results have been compared with solutions previously given.

3. *Potential theory*, by Professor M. G. Arsove, University of Washington.

From its inception at the end of the 18th century in the work of Legendre, Laplace, Gauss, Green, Dirichlet, and others, potential theory has been a powerful tool in the study of field problems of mathematical physics. Moreover, it has served as an important stimulus in the development of many branches of mathematical analysis. Classical potential theory had evolved so satisfactorily, in fact, that by the early 1900's its interest as a field of mathematical research had begun to wane. However, the advent of general measure theory and the pioneering work of F. Riesz in subharmonic functions were responsible for a renaissance of potential theory, this time as an instrument of pure mathematical research and with significant applications to complex function theory.

4. *The problem of difference sets*, by Professor T. G. Ostrom, Montana State University.

In this paper, the author traces the history of the difference set problem up to the most recent developments. A set of integers $\{a_0, a_1, \cdots, a_n\}$ is said to be a difference set mod N if the set of differences $\{a_i - a_j\}$ contains each non-zero residue mod N exactly once. The existence of a difference set mod N is equivalent to the existence of a cyclic projective plane with $n+1$ points on a line and a total of N points. In all known finite projective planes with $n+1$ points on a line, n is a prime power. The difference set problem is part of the more general problem for finite projective planes: "Must n be a prime power?"

5. *Mathematics in a women's college*, by Professor Andrewa R. Noble, Mills College.

In this paper the author discusses the courses, textbooks, requirements for the major, quality of students, and placement of graduates of the mathematics department at one college for women, Mills College, to show that the program differs little from that given in a coeducational institution of comparable size. The quality of students enrolled in mathematics classes, the size of classes, and light teaching loads are favorable factors at Mills College.

6. *The Math-Science Day at the University of Washington*, by Professor R. B. Leipnik, University of Washington.

A Spring Saturday in 1952 saw Math-Physics Day introduced on the University of Washington campus. About 1200 high-school students and 125 teachers attended from 70 schools. Though most "delegates" came from near Seattle, 20% came from over 100 miles. For students: job talks,

movies, research facilities inspection, numerous exhibits. For teachers: conferences, distribution of surplus (respectable!) college texts. Programs, identification cards and plastic Möbius strips were given each delegate. In 1953 Chemistry participated; the name became Math-Science Day and improved advance publicity produced greater attendance from outlying areas. Mathematics exhibits (including tic-tac-toe machine) were constructed in 300 man-hours (using Physics Shop facilities), being chosen for sturdiness, mathematical relevance and delegate operability.

7. The Symposium: *The Connecting Link Between College Algebra and Abstract Algebra*, moderated by Professor S. G. Hacker, State College of Washington.

Panel: (i) Professor Ivan Niven, University of Oregon.

In the first paper it was suggested that such a course have a reasonably self-contained program, sufficiently abstract to provide a basis for what is to follow and sufficiently problem-solving to emphasize the nature of proof. The speaker went on record as favoring the introduction and use of geometric or analytical concepts wherever they serve to improve or simplify the algebraic presentation. As outlined by him, the course should contain basic material in Theory of Equations with the emphasis on theory, Matrix Theory, including reduction of symmetric matrices under orthogonal transformation, and Group Theory.

(ii) Professor J. L. Brenner, State College of Washington.

In this paper the speaker urged that the course begin with Group Theory presented on a postulational basis. He stated that this was the first place in college mathematics where complete proofs were possible and he felt that there were advantages in striking out into new material at the beginning. He suggested that the course contain discussions of geometrical groups, analytic groups, group table, and special groups such as the two-group, four-group, etc. followed by more theoretical considerations. In connection with Lagrange's Theorem, he emphasized the importance of illustrating the theorem with examples.

(iii) Professor R. A. Beaumont, University of Washington.

In the concluding paper the following goals for such a junior-year algebra course were presented:

1) To teach facts about number systems, theory of equations and linear algebra, in the sense of a service course;

2) To introduce the principal concepts of modern algebra with the specific purpose of improving the first-year graduate course in algebra;

3) To develop an appreciation for precise mathematical statements and rigorous proofs.

He suggested beginning the course with linear algebra followed by material on groups, rings and fields and a systematic study of the number systems. The course is concluded with a discussion of polynomials, the theory of polynomial equations, and an introduction to the algebraic theory of fields.

J. M. KINGSTON, *Secretary.*

EMPLOYMENT OPPORTUNITIES

Beginning with the February 1954 issue, the MONTHLY will devote this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor, (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Washington) before the first day of the month preceding the issue in which the notice is to appear. Announcements should indicate the academic rank or similar description of the opening, but should not mention a specific salary. Blind ads are permissible which direct replies to a specific box number in care of the Mathematical Association of America, Buffalo 14, New York. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed by the Association at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.
ILLINOIS, Knox College, Galesburg, May 14–15, 1954.
INDIANA, Rose Polytechnic Institute, Terre Haute, May, 1954.
IOWA, Iowa State College, Ames, April, 1954.
KANSAS, Baker University, Baldwin City, March 27, 1954.
KENTUCKY
LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, February 19–20, 1954.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 5, 1953.
METROPOLITAN NEW YORK
MICHIGAN, University of Michigan, Ann Arbor, April, 1954.
MINNESOTA
MISSOURI, University of Missouri, Columbia, Spring, 1954.
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NORTHERN CALIFORNIA
OHIO, April, 1954.
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PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.
PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.
ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April, 1954.
SOUTHEASTERN, University of South Carolina, Columbia, March 12–13, 1954.
SOUTHERN CALIFORNIA, George Pepperdine College, Los Angeles, March 13, 1954.
SOUTHWESTERN, Arizona State College, Tempe.
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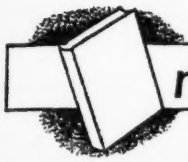

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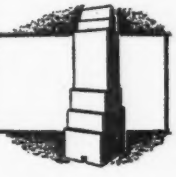

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